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THE LAMINAR COMPRESSIBLE BOUNDARY-LAYER IN THE STAGNATION-POINT REGION OF AN AXISYMMETRIC BLUNT BODY INCLUDING THE SECOND-ORDER EFFECT OF VORTICITY INTERACTION†

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Abstract—The influence of external vorticity on the laminar compressible boundary layer is examined in the stagnation-point region of an axisymmetric blunt body. The case considered is one where the vorticity is generated by a curved bow-shock wave formed by the body moving at a supersonic speed. The method of solution used is due to Van Dyke and consists of solving first-order (or Prandtl) boundary-layer equations and then solving second-order equations for the vorticity effect, where by second-order we mean second-order in a perturbation parameter related to the inverse square root of a Reynolds number.

The first step in the solution of both the first-order equations and the second-order equations for vorticity interaction is reduction of the partial differential equations to ordinary differential equations by a Blasius Series expansion. The resulting ordinary differential equations are then integrated numerically. Results are presented for integration of the equations for the first two terms of the series for the first-order equations and for one term of the series for the second-order equations. These results are obtained for a variety of wall to stagnation-point temperature ratios at the vertex. The results obtained from integrating these equations are compared with the theories and experiments of other authors. Good agreement is found with the theories of some other authors when a pressure gradient which is due to vorticity interaction is set equal to zero. It is found, however, that in general this simplification is not permissible. Agreement is also found with one set of experimental results, but this comparison is not very significant since the present analysis includes only the vorticity effect and not other second-order effects which may be equally important. However, since the second-order equations are linear the other second-order effects may be computed separately and superposed with the vorticity effect to obtain a complete second-order theory.

NOMENCLATURE

a,	nose radius;
a_1,	constant defined by equation (4.5c);
b_0, b_2,	constants defined by equation (2.3);
b_1,	constant defined by equation (4.4c);
C,	Sutherland constant defined by equation (1.1p);
C_p,	specific heat at constant pressure;
f_1, f_3,	functions defined by equation (2.5);
f_2,	function defined by equation (2.20);
g_1, g_3,	functions defined in general by equation (2.8) and for Sutherland's viscosity law by equations (2.34b, c);
h,	enthalpy;
H_1,	stagnation enthalpy;
j,	constant which equals 0 for plane flow and 1 for axisymmetric flow;

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M_∞, free stream Mach number;
n, co-ordinate normal to the body;
N, stretched co-ordinate normal to the body defined by equation (1.4);
p, pressure;
$P_1, P_2; p_1, p_2$, first- and second-order pressures in the outer and inner expansions, see equations (1.2c) and (1.3c);
q heat transfer;
q_1, q_2, first- and second-order heat transfer defined by equation (1.26);
r, body radius;
r_3, coefficient defined by equation (2.4);
R_1, R_2, first- and second-order density in the outer expansion, equation (1.2d);
R_∞, free stream Reynolds number defined by equation (1.1q);
Re_s, shock Reynolds number defined by equation (4.7a);
R_f, stagnation Reynolds number defined by equation (4.6a);
s, co-ordinate along the body;
S, entropy;
S_1, first-order entropy in the outer expansion;
T, absolute temperature;
$T_1, T_2; t_1, t_2$, first- and second-order temperatures in the outer and inner expansions, (1.2e) and (1.3e);
u, velocity component in the s direction;
U_∞, free stream velocity;
$U_1, U_2; u_1, u_2$, first- and second-order velocity components in the s direction in the outer and inner expansions, see equations (1.2a) and (1.3a);
v, velocity component in the n direction;
$V_1, V_2; v_1, v_2$, first- and second-order velocity components in the n direction in the outer and inner expansions, see equations (1.2b) and (1.3b);
w_1, w_3, constants defined by equations (2.2b) and (2.2c);
X_i, $i = 1, 2–5$, functions defined by equations (3.1 a–e);
X_i, $i = 6, 7–10$, functions defined by equations (3.3 a–e);
X_i, $i = 11, 12–15$, functions defined by equations (3.15 a–f);
X_{ij}, $i = 6, 7–10$, $j = 1, 2–4$, functions defined by equations (3.5 a–f).

Greek symbols

α, ratio of body to shock nose radius;
β_1, a quantity defined by equation (2.27e);
γ, ratio of specific heats C_p/C_v;
δ^*, displacement thickness defined by equation (1.31);
ϵ, perturbation parameter defined by equation (1.1r);
η, a co-ordinate normal to the body defined by equation (2·9);
μ, viscosity coefficient;
π_2, π_4, constants defined by equation (2.1);
ρ, density;
ρ_1, ρ_2, first- and second-order densities in the inner expansion, see equation (1.3d);
σ, Prandtl number; equation (1.29);
$\bar{\tau}$, shear stress defined by equation (1.29);
$\bar{\tau}_1, \bar{\tau}_2$, first- and second-order shear stress defined by equation (1.27);
τ_1, τ_3, functions defined by equation (2.7);
τ_2, function defined by equation (2.21);
ψ, stream function;

$\bar{\psi}_1, \bar{\psi}_2; \psi_1, \psi_2$, first- and second-order stream functions in the outer and inner expansions, see equations (1.2f) and (1.3f);

ω, exponent in the viscosity law where $\mu \alpha T^{\omega}$;

Ω_p vorticity interaction parameter defined by equation (4.1).

Subscripts

b, condition at the body surface;

o, stagnation point condition;

∞, free stream condition.

Superscripts

$*$, dimensional quantity;

$'$, differentiation with respect to the given argument.

INTRODUCTION

THE effect of external vorticity on the laminar boundary-layer in the stagnation-point region of an axisymmetric blunt body has been investigated by several authors. One of the more recent methods of investigation was that of Van Dyke [1] where the method of inner and outer expansions was used with a perturbation parameter related to the inverse square root of a Reynolds number for deriving the first- and second-order equations for the boundary-layer flow on blunt bodies. Van Dyke then used first terms of Blasius Series expansions in the co-ordinate along the body to find solutions to the first- and second-order equations in the stagnation-point region. In this present study Van Dyke's method is adopted and extended by using a viscosity law which varies as the square root of the temperature rather than linearly; in addition a variety of wall-temperature to stagnation-point temperature ratios are considered. Also included in this paper are solutions for the second term of the Blasius Series for the first-order boundary-layer equations. The use of the first two terms of the Blasius Series presented herein will give reasonably good results for boundary-layer computations for a region extending back to the sonic line on blunt bodies at supersonic speeds.

It will be shown that when the term involving the second-order pressure gradient along the body is set equal to zero, the results of integrating Van Dyke's second-order equation for the vorticity effect will agree with Probstein [2], Maslen [3], Lenard [19] and Cheng [4], but will not agree with Ferri, Zakkay and Ting [5]. However it will be shown that only at temperature ratios in the neighborhood of wall-temperature to stagnation-point temperature of 0·2 will the assumption of neglecting this pressure gradient be justifiable. Since for some time the only solution, which included the effect of the pressure gradient, was Van Dyke's for a temperature ratio of 0·2, wrong conclusions were drawn about the pressure gradient's effect. This happened because the displacement thickness for this particular temperature ratio is so small that its effect is negligible and therefore the second-order pressure gradient also becomes negligible.

Since the second-order equations are linear the pressure gradient due to vorticity interaction can be added to one of the other second-order terms such as the term due to displacement thickness. In most cases, however, where a complete second-order theory has been computed this has not been done and the term has been completely neglected. For a complete discussion of this and various ways to divide the second-order effects, see Van Dyke [20].

1. FORMULATION OF THE PROBLEM

Van Dyke [1] derived the first- and second-order boundary-layer equations for a plane or axisymmetrical blunt body at zero angle of attack. In deriving the equations he started from the full compressible Navier–Strokes and energy equations, and he used the systematic method of inner and outer expansions due to Lagerstrom, Kaplun and Cole. He has found that there are seven second-order effects, two of which are the effects of entropy and enthalpy gradients, or when taken together, the vorticity effect. We shall concern ourselves here only with the second-order effect of vorticity.

I. *Co-ordinate system*

A body of revolution lies in a flow field, with constant velocity U^*_∞ (parallel to the body axis) at *infinity*. The density ρ^*_∞ and temperature T^*_∞ are given. The specific heat C^*_p and Prandtl number σ are assumed to be constant, and the gas is assumed to be perfect. The co-ordinates of a point in the flow field and the velocity components are described by Fig. 1.

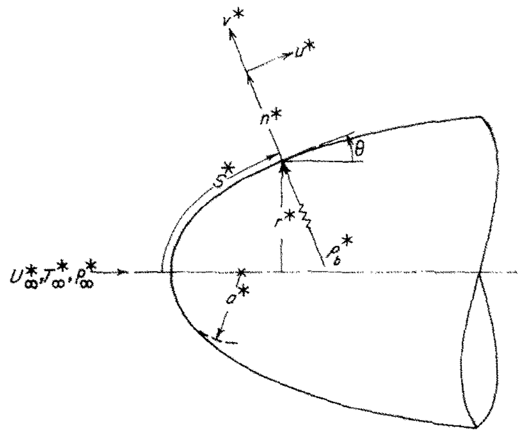


FIG. 1. Co-ordinate system.

II. *Dimensionless quantities*

In the following the unstarred quantities are the dimensionless and the starred quantities dimensional. The non-dimensional variables and constants remain bounded in the stagnation region as M_∞ goes to infinity. Co-ordinates and velocity components are explained in Fig. 1.

The quantity $j = 0$ for plane flow and 1 for axisymmetric flow.

$s = \dfrac{s^*}{a^*}$, co-ordinate along body surface; (1.1a)

$n = \dfrac{n^*}{a^*}$, co-ordinate normal to body surface; (1.1b)

$a = \dfrac{a^*}{a^*} = 1$ nose radius; (1.1c)

$u = \dfrac{u^*}{U^*_\infty}$, velocity component parallel to body surface; (1.1d)

$v = \dfrac{v^*}{U^*_\infty}$, velocity component normal to body surface; (1.1e)

$p = \dfrac{p^*}{\rho^*_\infty U^{*2}_\infty}$, pressure; (1.1f)

$\rho = \dfrac{\rho^*}{\rho^*_\infty}$, density; (1.1g)

$T = \dfrac{T^*}{U^{*2}_\infty/C^*_p}$ absolute temperature; (1.1h)

$\psi = \dfrac{\psi^*}{\rho^*_\infty U^*_\infty a^{*1+j}}$, stream function; (1.1i)

$S = \dfrac{S^*}{C^*_p}$, entropy; (1.1j)

$h = \dfrac{h^*}{U^{*2}_\infty}$, enthalpy; (1.1k)

$\mu = \dfrac{\mu^*(T^*)}{\mu^*(U^{*2}_\infty/C^*_p)}$, viscosity coefficient; (1.1l)

$\bar{\tau} = \dfrac{\bar{\tau}^*}{\rho^*_\infty U^{*2}_\infty}$, shear stress; (1.1m)

$q = \dfrac{q^*}{\rho^*_\infty U^{*3}_\infty}$, heat transfer; (1.1n)

$\bar{\delta}^* = \dfrac{\delta^*}{a^*}$, displacement thickness†; (1.1o)

$C = \dfrac{C^*}{T^*_\infty[1 + 1/2(\gamma - 1)M^2_\infty]}$, Sutherland constant; (1.1p)

$R_\infty = \dfrac{U^*_\infty a^* \rho^*_\infty}{\mu^*(T^*_\infty)}$, Reynolds number; (1.1q)

$\epsilon = [\mu^*(U^{*2}_\infty/C^*_p)/\rho^*_\infty U^*_\infty a^*]^{1/2}$, perturbation parameter. (1.1r)

Hereafter all unstarred quantities will be considered to be non-dimensionalized.

III. *Perturbation scheme*

Van Dyke [1] took the following expansion scheme with the perturbation parameter $\epsilon = [\mu^*(U^{*2}_\infty/C^*_p)/\rho^*_\infty U^*_\infty a^*]^{1/2}$. This scheme seems reasonable as long as the body is analytic. We could have first left the dependence on ϵ unspecified in the expansion, and upon substitution into the full Navier–Stokes and energy equations we would have found that when we consider the boundary conditions the only

† The displacement thickness is usually divided by a Reynolds number in non-dimensionalizing it, which is not done here.

meaningful expansion scheme is the one given below.

Outer expansion

$$u(s, n; \epsilon) \sim U_1(s, n) + \epsilon U_2(s, n) + \ldots \dagger \quad (1.2a)$$
$$v(s, n; \epsilon) \sim V_1(s, n) + \epsilon V_2(s, n) + \ldots \quad (1.2b)$$
$$p(s, n; \epsilon) \sim P_1(s, n) + \epsilon P_2(s, n) + \ldots \quad (1.2c)$$
$$\rho(s, n; \epsilon) \sim R_1(s, n) + \epsilon R_2(s, n) + \ldots \quad (1.2d)$$
$$T(s, n; \epsilon) \sim T_1(s, n) + \epsilon T_2(s, n) + \ldots \quad (1.2e)$$
$$\psi(s, n; \epsilon) \sim \bar{\psi}_1(s, n) + \epsilon \bar{\psi}_2(s, n) + \ldots . \quad (1.2f)$$

Inner expansion

$$u(s, n; \epsilon) \sim u_1(s, N) + \epsilon u_2(s, N) + \ldots \quad (1.3a)$$
$$v(s, n; \epsilon) \sim \epsilon v_1(s, N) + \epsilon^2 v_2(s, N) + \ldots \quad (1.3b)$$
$$p(s, n; \epsilon) \sim p_1(s, N) + \epsilon p_2(s, N) + \ldots \quad (1.3c)$$
$$\rho(s, n; \epsilon) \sim \rho_1(s, N) + \epsilon \rho_2(s, N) + \ldots \quad (1.3d)$$
$$T(s, n; \epsilon) \sim t_1(s, N) + \epsilon t_2(s, N) + \ldots \quad (1.3e)$$
$$\psi(s, n; \epsilon) \sim \epsilon \psi_1(s, N) + \epsilon^2 \psi_2(s, N) + \ldots . \quad (1.3f)$$

In the above expressions the normal co-ordinates are related by

$$N = \frac{n}{\epsilon}. \quad (1.4)$$

It is also necessary to expand the viscosity in a Taylor series expansion about t_1 as follows:

$$\mu(T) = \mu(t_1) + \epsilon \mu'(t_1) t_2 + \ldots \quad (1.5)$$

with

$$\mu'(t_1) = \left(\frac{\partial \mu}{\partial t}\right)_{t_1}. \quad (1.6)$$

The perturbation parameter ϵ can also be expressed as

$$\epsilon = \frac{1}{\sqrt{(R_\infty)}} \left\{ \frac{\mu[(\gamma - 1) M_\infty^2 T_\infty^*]}{\mu(T_\infty^*)} \right\}^{1/2}. \quad (1.7)$$

If it is assumed that the viscosity coefficient μ is proportional to T to some power ω then ϵ, the perturbation parameter, can be written as follows.

$$\epsilon = \frac{[(\gamma - 1) M_\infty^2]^{\omega/2}}{\sqrt{(R_\infty)}}. \quad (1.8)$$

Here M_∞ is the free stream Mach number, and R_∞ the Reynolds number formed with the nose radius as reference length.

† The symbol $\sim$ means asymptotically equal to.

In all of the work contained in this report the definition of ϵ used will be that of equation (1.8) and ω will be taken to be 1/2.

The expansion schemes used herein seem reasonable since the boundary-layer thickness at high Mach numbers can be shown to be $0(\epsilon)$ whereas the thickness of the bow shock wave is $0(\epsilon^2)$ and the shock-layer thickness itself is 0(1).

Implicitly contained in the above is the assumption that the viscosity varies as the absolute temperature to some power ω, for example when ω is taken to be 1 this would be the linear-viscosity law. This assumption is not necessary, and a more exact viscosity law such as Sutherland's could be used. Later a Blasius Series will be used for solutions near the stagnation point. By using the power-viscosity law, Blasius Series solutions in the boundary layer can be obtained by knowing only the ratio of the temperatures across the boundary layer. When Sutherland's law is used, however, solutions depend not only on the temperature ratio across the boundary layer but on the free-stream conditions. This means that if the Sutherland viscosity law is employed we cannot tabulate solutions for different wall-temperature to stagnation-point temperature ratios, but must integrate the differential equations resulting from the use of the Blasius Series every time the free-stream conditions are changed.

IV. *Boundary conditions*

The proper boundary conditions for the Navier–Stokes and energy equations when slip and temperature jump at the body surface are neglected, are:

Conditions at the body

$$u(s, 0) = 0,\ v(s, 0) = 0,\ T(s, 0) = T_b(s). \quad (1.9a, b, c)$$

Upstream conditions

$$\left.\begin{aligned} \mathbf{u} + \mathbf{v} &\to \mathbf{i} \quad &(1.9d)\\ p &\to \frac{1}{\gamma M_\infty^2} \quad &(1.9e)\\ \rho &\to 1 \quad &(1.9f)\end{aligned}\right\} \text{as } n \to \infty.$$

Where $\mathbf{u}$ and $\mathbf{v}$ are taken to be the velocity vectors and $\mathbf{i}$ is a unit vector parallel to the body axis in the streamwise direction.

V. *Matching conditions*

The inner expansion is valid in a region of $0(\epsilon)$ near the body and the outer expansion is valid in the region outside this region of $0(\epsilon)$. In substituting the inner and outer expansions into the Navier–Stokes and energy equations the resulting partial differential equations from taking successive terms in ϵ are of lower order than the original equations. This means that we cannot in general expect the resulting equations to satisfy all of the boundary conditions which the original Navier–Stokes and energy equations satisfied. For instance we do not expect the outer expansion to satisfy the condition of zero u component of velocity at the wall, or if slip is permitted the slip condition will be violated. This means that the "lost" boundary conditions must be replaced by something which makes the problem determinate. These conditions are found to be the matching principle of Lagerstrom [6]. A good explanation of the matching principle along with a more rigorous discussion of inner and outer expansions in general can also be found in a paper by Erdelyi [12].

The matching principle can be stated as follows:

m-term inner expansion of (p-term outer expansion)
$= p$-term outer expansion of (m-term inner expansion).

Van Dyke [1] applied this principle with $m = p$ and $m = p - 1$ and obtained the appropriate matching conditions.

This way of applying the matching principle perhaps obscures what is actually taking place in the matching process, and a more simple procedure is possible. It is desired to find the appropriate values which the terms in the outer expansion should approach at the wall and the values which the terms in the inner expansion should approach as N, the inner variable, goes to infinity. The matching principle supposes that there is a relationship between these two behaviors, or an overlap region. If we suppose this then the following should seem reasonable.

Suppose we have some quantity in the flow field, say v the component of velocity normal to the wall. Then in the outer region v is given as follows from equation (1.2b):

$$v \sim V_1(s, n) + \epsilon V_2(s, n) + \ldots \quad (1.10a)$$

In the inner region v is given as follows from equation (1.3b).

$$v \sim \epsilon\, v_1(s, N) + \epsilon^2 v_2(s, N) + \ldots \quad (1.10b)$$

There are two ways in which we will match these quantities. The first is performed in the overlap region or the region in the inner variable where $N \to \infty$. This is done as follows:

The inner expansion in this region simply becomes:

$$v \sim \epsilon\, v_1(s, N) + \epsilon^2 v_2(s, N) + \ldots \text{ as } N \to \infty. \quad (1.10c)$$

For the outer expansion we must expand the terms in a series about $n = 0$ as follows:

$$v \sim V_1(s, 0) + nV_{1n}(s, 0) + (n^2/2) V_{1nn}(s, 0) + \ldots + \epsilon V_2(s, 0) + \ldots \quad (1.10d)$$

or in inner variables

$$v \sim V_1(s, 0) + \epsilon N V_{1n}(s, 0) + \epsilon V_2(s, 0) + \ldots \quad (1.10e)$$

Therefore in the overlap region of the outer and inner expansions we have for the outer expansion

$$v \sim V_1(s, 0) + \epsilon N V_{1n}(s, 0) + \epsilon V_2(s, 0) + \ldots \text{ as } N \to \infty. \quad (1.10f)$$

Therefore since we have assumed that the behavior of these functions are the same in the overlap region we have from equating coefficients of equal powers of ϵ the following:

$$V_1(s, 0) = 0 \quad (1.10g)$$

$$v_1(s, N) \sim N V_{1n}(s, 0) + V_2(s, 0) \text{ as } N \to \infty. \quad (1.10h)$$

This is exactly the same result as obtained by Van Dyke [1], equation (2.40b). This procedure gives a result equivalent to taking $m = p$ in the matching principle.

Next we can match in a similar manner by expanding the inner expansion to $N = 0$ in an asymptotic series about a large value of N. We then equate this to the value of the outer expansion at $n = 0$. We have again assumed here that the inner expansion for large N behaves in the same manner as the outer expansion for small n.

For the inner expansion we have equation (1.10b). Therefore expanding to $N = 0$ about a large value of N (let us say N_o) in an asymptotic series gives:

$$v \sim \epsilon\,[v_1(s, N_o) - N_o\, v_{1N}(s, N_o)] + \dots \quad \text{as } N_o \to \infty. \quad (1.10\text{i})$$

We notice that in the asymptotic series expansion for $v_1(s, N)$ above only two terms are needed since $v_{1NN}(s, N)$ and all higher derivatives in N are exponentially small as N goes to infinity.

From the outer expansion we get at $n = 0$

$$v \sim V_1(s, 0) + \epsilon\, V_2(s, 0) + \dots \quad (1.10\text{j})$$

Equating coefficients of equal powers of ϵ we get as before

$$V_1(s, 0) = 0 \quad (1.10\text{k})$$

and in addition

$$V_2(s, 0) = \lim_{N\to\infty} [v_1(s, N) - N v_{1N}(s, N)] \quad (1.10\text{l})$$

which is Van Dyke's equation (2.39) and is equivalent to the result of using the matching principle for $m = p - 1$.

This matching principle is perhaps the most crucial part of the analysis since the use of it definitely gives a second-order pressure gradient due to vorticity interaction as will be seen later.

Still another, and the most physical way of seeing the matching of the v components is given as follows by using the stream function.

Physically we visualize a blunt body in a flow field. To the first approximation there is an inviscid flow around the body with a boundary layer (first order). Then in the second approximation the boundary layer increases the thickness of the body and the outer flow sees a body which is thickened by this displacement thickness. Then to the second approximation in the outer flow, the stream function should vanish at the surface of a body which consists of the original body plus displacement thickness. As will be seen later in equation (1.31) the displacement thickness can be written as

$$\delta^* = \epsilon \int_0^\infty \left[1 - \frac{\rho_1 u_1}{(R_1 U_1)_{n=0}}\right] \mathrm{d}N. \quad (1.10\text{m})$$

Y

In terms of the first-order stream function in the inner region this becomes (see 1.3f)

$$\delta^* = \epsilon \lim_{N\to\infty} \left[N - \frac{\psi_1(s, N)}{\psi_{1N}(s, N)}\right] \quad (1.10\text{n})$$

where

$$\psi_{1N} = r^j \rho_1 u_1 \text{ and } \psi_{1s} = -r^j \rho_1 v_1 \quad (1.10\text{o})$$

define the stream function.

Then applying the boundary condition for $\bar{\psi}(s, n)$ at the outer edge of the new body given by the original body plus displacement thickness we have

$$\bar{\psi}(s, \delta^*) = 0 = \bar{\psi}_1(s, \delta^*) + \epsilon\,\bar{\psi}_2(s, \delta^*). \quad (1.10\text{p})$$

It is understood that the zero appearing above means zero only to terms of order ϵ.

Now expanding the above expression in a Taylor series about the old body surface we get

$$0 = \bar{\psi}_1(s, 0) + \delta^*\,\bar{\psi}_{1n}(s, 0) + \dots + \epsilon\,\bar{\psi}_2(s, 0) + \dots. \quad (1.10\text{q})$$

Using the relation for δ^* we have

$$\bar{\psi}_1(s, 0) = 0 \quad (1.10\text{r})$$

$$\bar{\psi}_2(s, 0) = \lim_{N\to\infty} \left[\psi_1(s, N)\frac{\bar{\psi}_{1n}(s, 0)}{\psi_{1N}(s, N)} - N\,\bar{\psi}_{1n}(s, 0)\right]. \quad (1.10\text{s})$$

Now we use the fact that

$$\bar{\psi}_{1n}(s, 0) = \psi_{1N}(s, N) \text{ as } N \to \infty$$

which has been known since the boundary-layer theory of Prandtl or is known from matching.

Therefore as a final result we have

$$\bar{\psi}_2(s, 0) = \lim_{N\to\infty} [\psi_1(s, N) - N\,\psi_{1N}(s, N)] \quad (1.10\text{t})$$

or differentiating with respect to s and using some other relations we can show that

$$V_2(s, 0) = \lim_{N\to\infty} [v_1(s, N) - N\, v_{1N}(s, N)]. \quad (1.10\text{u})$$

The analysis above is not very complete and reasoning such as this can lead to serious error, but in this case it does give some further physical insight into what the matching principle gives.

VI. *First- and second-order equations for the boundary layer*

First the full compressible Navier–Stokes and energy equations are written in the co-ordinate system of Part 1, non-dimensionalized by equations (1.1a–r) and expanded first into the outer expansion and then into the inner expansion by equations (1.2a–e) and equations (1.3a–e). Then by collecting terms in successive powers of ϵ and equating these to zero we obtain the partial differential equations describing the outer and inner flows. By simplifying these results we can obtain the following equations. They comprise both plane and axisymmetric flow. The exponent j equals 0 for plane flow and equals 1 for axisymmetric flow. The subscripts s and N indicate differentiation, and S_1' denotes $dS_1/d\bar{\psi}_1$ where S_1 and $\bar{\psi}_1$ are the first-order entropy and stream functions respectively in the outer inviscid flow. (The conventional entropy and stream function obtained from the compressible Euler equations.)

FIRST-ORDER BOUNDARY-LAYER EQUATIONS

Differential equations

Continuity

$$(r^j \rho_1 u_1)_s + (r^j \rho_1 v_1)_N = 0. \tag{1.11}$$

Momentum

$$\rho_1 (u_1 u_{1s} + v_1 u_{1N}) - (\mu u_{1N})_N = (R_1 U_1 U_{1s})_{n=0}. \tag{1.12}$$

Energy

$$\rho_1 \left(u_1 \frac{\partial}{\partial s} + v_1 \frac{\partial}{\partial N}\right)\left(t_1 + \frac{1}{2}u_1^2\right) - \frac{\partial}{\partial N}\left[\mu \left(\sigma^{-1} t_1 + \frac{1}{2}u_1^2\right)_N\right] = 0. \tag{1.13}$$

Pressure condition

$$\rho_1 t_1 = (R_1 T_1)_{n=0}. \tag{1.14}$$

Boundary conditions

$$u_1(s, 0) = v_1(s, 0) = 0 \qquad (1.15\text{a, b})$$

$$t_1(s, 0) = T_b(s) \tag{1.15c}$$

or a condition on the wall heat transfer.

Matching conditions

$$\left.\begin{aligned} u_1(s, N) &\sim U_1(s, 0) \qquad (1.16\text{a})\\ t_1(s, N) &\sim T_1(s, 0) \qquad (1.16\text{b}) \end{aligned}\right\} \text{ as } N \to \infty.$$

These are the familiar compressible boundary-layer equations in non-dimensional form.

SECOND-ORDER BOUNDARY-LAYER EQUATIONS INCLUDING THE EFFECT OF VORTICITY DUE TO ENTROPY GRADIENT ONLY

Differential Equations

Continuity

$$[r^j(\rho_1 u_2 + \rho_2 u_1)]_s + [r^j(\rho_1 v_2 + \rho_2 v_1)]_N = 0. \tag{1.17}$$

Momentum

$$\rho_1 (u_1 u_{2s} + u_2 u_{1s} + v_1 u_{2N} + v_2 u_{1N}) + \rho_2 (u_1 u_{1s} + v_1 u_{1N}) - (\mu u_{2N} + \mu' u_{1N} t_2)_N = -r^j (S_1' R_1^2 T_1 V_2)_{n=0}. \tag{1.18}$$

Energy

$$\rho_1 \left(u_1 \frac{\partial}{\partial s} + v_1 \frac{\partial}{\partial N}\right)(t_2 + u_1 u_2) + \left[\rho_1 \left(u_2 \frac{\partial}{\partial s} + v_2 \frac{\partial}{\partial N}\right) + \rho_2 \left(u_1 \frac{\partial}{\partial s} + v_1 \frac{\partial}{\partial N}\right)\right]\left(t_1 + \frac{1}{2}u_1^2\right) - \frac{\partial}{\partial N}\left[u(\sigma^{-1} t_2 + u_1 u_2)_N + \mu' \left(\sigma^{-1} t_1 + \frac{1}{2}u_1^2\right)_N t_2\right] = 0. \tag{1.19}$$

Pressure condition

$$\rho_1 t_2 + \rho_2 t_1 = 0. \tag{1.20}$$

Boundary conditions

$$u_2(s, 0) = v_2(s, 0) = t_2(s, 0) = 0. \qquad (1.21\text{a, b, c})$$

Matching conditions

$$\left.\begin{aligned} u_2(s, N) &\sim -N r^j (S_1' R_1 T_1)_{n=0} \qquad (1.22\text{a})\\ t_2(s, N) &\sim N r^j (S_1' R_1 T_1 U_1)_{n=0} \qquad (1.22\text{b}) \end{aligned}\right\} \text{ as } N \to \infty.$$

The term $S_1'(0)$ appearing in the above equations is defined as below:

$$S_1'(0) = \frac{dS_1(0)}{d\bar{\psi}_1} = -\frac{4(\gamma - 1)(M_\infty^2 - 1)^2 \alpha^{1+j}}{[2\gamma M_\infty^2 - (\gamma - 1)][2 + (\gamma - 1) M_\infty^2]} 0^{1-j} \tag{1.23}$$

where $0^{1-j} = 0$ for plane flow and 1 for axisymmetric flow. α is the ratio of body to shock nose radius. Therefore we see that there is no effect of vorticity for plane flow to the second order.

There is also an effect of vorticity due to $H_1'(0)$, the enthalpy gradient. However in flows where the enthalpy is constant throughout the flow field in front of the shock wave, it will remain constant across the shock wave and $H_1'(0) \equiv 0$. This is the usual case and we will therefore not consider this effect here.

The term on the right-hand side of the second-order momentum equation (1.18) is actually the second-order pressure gradient due to vorticity since it can be shown that

$$P_{2s} = r^j [R_1^2 \, T_1 S_1' V_2]_{n=0} \qquad (1.24)$$

Various authors have said that this effect does not exist or that it is negligible. In the numerical examples (Fig. 16) it will be shown that this effect is not only important, but also that the second-order terms in heat transfer and skin friction obtained by neglecting it can be off by a factor of 2 or more.†

Appearing on the right-hand side of the second-order momentum equation (1.18) is the quantity $V_2(s, 0)$. It has been shown in equation (1.101) that in applying the matching principle,

$$V_2(s, 0) = \lim_{N \to \infty} (v_1 - N v_{1N}). \qquad (1.25)$$

This means that we do not need to solve the second-order problem for the outer flow in order to calculate the second-order effect of vorticity in the boundary-layer. As would be expected the first-order equations for the outer flow do not show a viscosity influence and therefore turn out to be the familiar compressible Euler equations.

VII. *Heat transfer, shear stress, and displacement thickness*

The non-dimensional heat transfer and shear stress are defined as follows:

Heat transfer

$$q = -\frac{1}{\rho_\infty^* U_\infty^{*3}} \frac{C_p^*}{\sigma} \mu^* \frac{\partial T^*}{\partial n^*} = \epsilon\, q_1 + \epsilon_2 q^2 + \dots \qquad (1.26)$$

† Added in proof: This pressure gradient term could be included instead in the term for the effect of displacement thickness, however most authors fail to include it in either place.

Shear stress

$$\bar{\tau} = \frac{1}{\rho_\infty^* U_\infty^{*2}} \mu^* \frac{\partial u^*}{\partial n^*} = \epsilon\, \bar{\tau}_1 + \epsilon^2\, \bar{\tau}_2 + \dots \qquad (1.27)$$

In terms of the first- and second-order boundary-layer quantities the non-dimensional heat transfer and shear stress are then found to be as follows:

Heat transfer

$$q = -\epsilon \frac{1}{\sigma} \mu(t_1) \frac{\partial t_1}{\partial N} - \epsilon^2 \frac{1}{\sigma} \left[\mu(t_1) \frac{\partial t_2}{\partial N} + \mu'(t_1)\, t_2 \frac{\partial t_1}{\partial N} \right] + \dots \qquad (1.28)$$

Shear stress

$$\bar{\tau} = \epsilon \mu \frac{\partial u_1}{\partial N} + \epsilon^2 \left[\mu \frac{\partial u_2}{\partial N} + \mu'(t_1)\, t_2 \frac{\partial u_1}{\partial N} \right] + \dots \qquad (1.29)$$

The displacement thickness for the first-order boundary layer is defined as follows:

$$\delta^* = \int_0^\infty \left[1 - \frac{\rho_1^* u_1^*}{(R_1^* U_1^*)} \right]_{n=0} dn^*. \qquad (1.30)$$

Using non-dimensional quantities we therefore get:

$$\bar{\delta}^* = \epsilon \int_0^\infty \left[1 - \frac{\rho_1 u_1}{(R_1 U_1)} \right]_{n=0} dN. \qquad (1.31)$$

2. SOLUTION OF THE FIRST- AND SECOND-ORDER BOUNDARY-LAYER EQUATIONS

There are few exact solutions known to the first-order compressible boundary-layer equations. These solutions are for either very special flow conditions where the governing partial differential equations can be solved directly or for conditions where similar solutions can be obtained by reducing the partial differential equations to ordinary differential equations and integrating these equations numerically. Some of these solutions could be extended to the second-order boundary-layer problem, however since we are interested in solving the problem of the boundary-layer on an axisymmetric blunt body of arbitrary shape it seems advisable to proceed in a manner which will allow us to solve the equations when arbitrary boundary and external conditions are given. This suggests the use of a numerical method of solution, such as a modification of the finite difference method given by Flügge-Lotz and Blottner [8]. Their method was developed for the plane case only, but it could be easily modified to the axisymmetric case. In order to use such a method it is necessary to find velocity and temperature profiles near the stagnation-point for starting values for the finite difference scheme. It seems that an ideal method of finding these starting profiles is the use of a Blasius Series as given by Van Dyke [1] where the velocities, temperatures, etc. are expanded in power series along the body and the resulting ordinary differential equations are integrated numerically. The first term of these series will give the solution near the stagnation point. Second and third terms in the series will give greater accuracy for the computation of flow quantities away from the stagnation point. The Blasius Series may converge in the entire subsonic region, however since we are interested in calculating quantities in the boundary-layer at points on the body further back than the region of convergence of the Blasius Series it seems advisable to start with the finite difference scheme as near the stagnation-point as possible. We will therefore find solutions to the problem for the first two terms of the Blasius Series for the first-order boundary-layer equations, and the first term of the series for the second-order effect of vorticity with the idea of later using these results to proceed downstream with a finite-difference method.

I. *Mangler, Howarth, Dorodnitsyn transformation*

Hayes and Probstein ([2], p. 290) have used a transformation which includes the Mangler and Howarth–Dorodnitsyn transformations to transform the first-order boundary-layer equations into a form where similar solutions can be easily obtained. A Blasius Series (p. 322) is then used in the transformed variables to obtain ordinary differential equations for the solution of the blunt body problem. The same procedure could be used here for both the first- and second-order equations and solutions in the transformed variables could be obtained, however it seems that it would be advisable to stay as near the physical variables as possible. This is especially important when it is intended to use the results as starting values for solving the boundary-layer equations by the finite-difference method of Flügge-Lotz and Blottner [8]. Their method I uses the physical variables and method II uses the Howarth–Dorodnitsyn transformed variables. This means that if the physical variables are used for the stagnation-point solutions it will be much easier to go directly to method I and will also be easy to use method II. It is also desirable to stay as close to the physical variables as possible since it is easier to see how the flow quantities are behaving in physical variables rather than visualizing them in the transformed variables. We might also consider using the Mangler transformation to transform the axisymmetric flow case to an equivalent two-dimensional flow problem. However, the use of the transformation is disappointing, since the resulting pressure gradient on the transformed blunt body, a transformed sphere for example, goes to infinity like $1/\bar{s}^{1/3}$ as $\bar{s} \rightarrow 0$ where $\bar{s}$ is the Mangler transformed variable. This means that the use of a finite difference method near the stagnation point of the transformed body would prove difficult. The Mangler transformation also does not offer any advantage in solving the equations by a Blasius Series method and therefore there seems to be no reason for using it. Another reason for using the physical variables is that if transformations of the type given by Hayes and Probstein [2] are used on the second-order equations (1.17–1.20), the resulting expressions become very complicated. For these reasons the expansions used will be obtained from expanding directly in the physical dimensionless variables.

II. *Blasius Series for the solution of the first-order boundary-layer equations*

Following Van Dyke [1] we expand the flow quantities in a Blasius Series. The choice of odd and even powers of s in the expansion can be explained by observing that the u component of velocity must be an odd function of s whereas all other flow quantities must be even functions of s. Subscripts 10 mean dimensionless inviscid stagnation-point quantities, i.e. P_{10} is the dimensionless inviscid stagnation-point pressure.

The expansions for the outer flow quantities evaluated at the body are as follows:

$$P_1(s,0) = P_{10}(1 - \pi_2 s^2 - \pi_4 s^4 + \ldots) \quad (2.1)$$

$$U_1(s,0) = w_1 s + w_3 s^3 + \ldots \quad (2.2a)$$

where it can be shown that

$$w_1 = \left(2\frac{\gamma-1}{\gamma}T_{10}\,\pi_2\right)^{1/2} \quad (2.2b)$$

and

$$w_3 = \frac{w_1}{2\pi_2}\left(\frac{\pi_2^2}{2\gamma} + \pi_4\right). \quad (2.2c)$$

The body temperature and radius are expanded as follows:

$$T_b(s) = T_{10}(b_0 + b_2 s^2 + \ldots) \quad (2.3)$$

$$r^j(s) = s^j(1 + j\,r_3 s^2 + \ldots). \quad (2.4)$$

By introducing the stream function we can satisfy the continuity equation (1.11). The Blasius Series expansion for the first-order term of the stream function is as follows: (see equation 1.3f)

$$\psi_1(s,N) = \left(\frac{R_{10}\mu_{10}}{w_1}\right)^{1/2} s^{1+j}\,[w_1 f_1(\eta) + w_1\pi_2 f_3(\eta)s^2 + \ldots]. \quad (2.5)$$

Where the stream function satisfies the following relations:

$$\psi_{1N} = r^j\rho_1 u_1,\ \psi_{1s} = -r^j\rho_1 v_1. \quad (2.6)$$

The expansions for the temperature and viscosity are as follows:

$$t_1(s,N) = T_{10}[\tau_1(\eta) + \pi_2\tau_3(\eta)s^2 + \ldots] \quad (2.7)$$

$$\mu(t_1) = \mu_{10}[g_1(\eta) + \pi_2 g_3(\eta)s^2 + \ldots]. \quad (2.8)$$

Where in all of the above expressions η is defined as

$$\eta = \left[\frac{w_1 R_{10}}{\mu_{10}}\right]^{1/2} N. \quad (2.9)$$

From the relations (2.1) and (2.7) we find that the expansion for the density has the form

$$\rho_1 = R_{10}\left[\frac{1}{\tau_1} - \frac{\pi_2}{\tau_1}\left(\frac{\tau_3}{\tau_1} + 1\right)s^2 + \ldots\right]. \quad (2.10)$$

Expansions for the velocity components, shear stress, heat transfer and displacement thickness follow easily and are given below:

$$u_1 = w_1\tau_1 f_1' s + w_1\pi_2\left[\tau_1 f_3' + \tau_1 f_1'\left(\frac{\tau_3}{\tau_1} + 1 - j\frac{r_3}{\pi_2}\right)\right]s^3 + \ldots \quad (2.11)$$

$$v_1 = -\left(\frac{\mu_{10}}{R_{10}w_1}\right)^{1/2}\left\{w_1(j+1)\tau_1 f_1 + w_1\pi_2\left[(j+3)\tau_1 f_3 + (j+1)\tau_1 f_1\left(\frac{\tau_3}{\tau_1} + 1 - j\frac{r_3}{\pi_2}\right)\right]s^2 + \ldots\right\} \quad (2.12)$$

$$\bar{\tau}_1 = (2w_1\pi_2 P_{10}\mu_{10})^{1/2}\left(g_1(\tau_1 f_1')'s + \pi_2\left\{g_3(\tau_1 f_1')' + g_1\left[\tau_1 f_3' + \tau_1 f_1'\left(\frac{\tau_3}{\tau_1} + 1 - j\frac{r_3}{\pi_2}\right)\right]\right\}s^3 + \ldots\right) \quad (2.13)$$

$$q_1 = -\frac{T_{10}}{\sigma}(w_1 R_{10}\mu_{10})^{1/2}\,[g_1\tau_1' + \pi_2(g_1\tau_3' + g_3\tau_1')s^2 + \ldots] \quad (2.14)$$

$$\delta^* = \epsilon\left(\frac{\mu_{10}}{w_1 R_{10}}\right)^{1/2}\lim_{\eta\to\infty}\left\{\eta - f_1 - \pi_2\left[f_3 - \left(\frac{\pi_4}{2\pi_2^2} - \frac{3}{4\gamma} + j\frac{r_3}{\pi_2}\right)f_1\right]s^2 + \ldots\right\}. \quad (2.15)$$

The limit as $\eta \to \infty$ of $\eta - f_1$ in the displacement thickness expression also appears later in the second-order equations, and will remain finite in the limit.

Substituting expressions (2.7)–(2.12) into equations (1.11)–(1.16) we obtain the ordinary differential equations for the first two terms of the series.

First terms

$$[g_1(f_1'\tau_1)']' + (j+1)f_1(\tau_1 f_1')' - \tau_1 f_1'^2 = -1 \quad (2.16a)$$

$$[g_1\tau_1']' + \sigma(j+1)\tau_1' f_1 = 0 \quad (2.16b)$$

$$f_1(0) = 0,\ f_1'(0) = 0,\ \tau_1(0) = b_o \quad (2.16c)$$

$$f_1'(\infty) = 1,\ \tau_1(\infty) = 1 \quad (2.16d)$$

Second terms

$$[g_3(f_1'\tau_1)' + g_1(f_1'\tau_3 + f_3'\tau_1)']' - 3f_1'^2\tau_3 - 4f_1'\tau_1 f_3' + (j+3)(f_1'\tau_1)'f_3 + (j+1)f_1(f_3'\tau_1 + f_1'\tau_3)' = 1 - 2\frac{\pi_4}{\pi_2^2} + 2\tau_1 f_1'^2 - j\frac{r_3}{\pi_2}[3f_1'^2\tau_1 - 2f_1(f_1'\tau_1)' + 1] \quad (2.17a)$$

$$\frac{1}{\sigma}[g_3\tau_1' + g_1\tau_3']' - 2f_1'\tau_3 + (j+1)f_1\tau_3' + (j+3)\tau_1'f_3 = 2\left(\frac{\gamma-1}{\gamma}\right)[f_1'\tau_1 - g_1(f_1'\tau_1)'^2] + 2j\frac{r_3}{\pi_2}f_1\tau_1' \quad (2.17b)$$

$$f_3(0) = 0, f_3'(0) = 0, \tau_3(0) = \frac{b_2}{\pi_2} \quad (2.17c)$$

$$\left.\begin{aligned} f_3'(\infty) &= -\frac{3}{4\gamma} + \frac{\pi_4}{2\pi_2^2} + j\frac{r_3}{\pi_2} \\ \tau_3(\infty) &= -\left(\frac{\gamma-1}{\gamma}\right). \end{aligned}\right\} \quad (2.17d)$$

The equations for the first terms represent the flow in the region near the stagnation point. The second terms allow greater accuracy away from the stagnation point, and higher accuracy is obtained by using third and higher terms. Integration will be carried out for the first and second terms only. The equations for the first term are non-linear and will present the greatest difficulty in integration. The equations for the second terms are linear and as in the incompressible case can be divided into universal functions and integrated once and for all once b_0 is chosen in the equation for the first term. (See for instance Schlichting [9], p. 185 for the incompressible case).

III. *Blasius Series for the solution of the second-order boundary-layer equations for the effect of vorticity*

In a manner similar to Section II we expand the second-order equations. The second-order term in the stream function, $\psi_2(s, N)$, satisfies the second-order continuity equation (1.17) by taking

$$\psi_{2N} = r^j(\rho_1 u_2 + \rho_2 u_1) \quad (2.18)$$

$$\psi_{2s} = -r^j(\rho_1 v_2 + \rho_2 v_1). \quad (2.19)$$

Then for the same reasons as in the first-order equations (Section II, paragraph 1), we take

$$\psi_2(s, N) = \frac{R_{10}T_{10}\mu_{10}}{w_1}S_1'(0)\,s^{1+j}[f_2(\eta) + \ldots]. \quad (2.20)$$

$$t_2(s, N) = \frac{T_{10}^2}{w_1}S_1'(0)\left(\frac{R_{10}\mu_{10}}{w_1}\right)^{1/2}[\tau_2(\eta) + \ldots]. \quad (2.21)$$

From equations (2.21), (2.7) and (2.10) along with (1.20) we find the expansion for ρ_2

$$\rho_2 = -\frac{R_{10}T_{10}S_1'(0)}{w_1}\left(\frac{R_{10}\mu_{10}}{w_1}\right)^{1/2}\left[\frac{\tau_2}{\tau_1^2} + \ldots\right]. \quad (2.22)$$

Equations (2.18) and (2.19) yield the expansions for the second-order velocity terms

$$u_2 = T_{10}S_1'(0)\left(\frac{R_{10}\mu_{10}}{w_1}\right)^{1/2}[(\tau_1 f_2' + \tau_2 f_1')s + \ldots] \quad (2.23)$$

$$v_2 = -\frac{T_{10}\mu_{10}S_1'(0)}{w_1}(1+j)[(\tau_1 f_2 + \tau_2 f_1) + \ldots]. \quad (2.24)$$

Second-order shear stress and heat-transfer effects are given by

$$\bar{\tau}_2 = \mu_{10}R_{10}T_{10}S_1'(0)\left\{\left[g_1(\tau_1 f_2' + \tau_2 f_1')' + \frac{dg_1}{d\tau_1}\tau_2(\tau_1 f_1')'\right]s + \ldots\right\} \quad (2.25)$$

$$q_2 = -\frac{1}{\sigma}\frac{R_{10}\mu_{10}}{w_1}T_{10}^2 S_1'(0)\left\{\left[g_1\tau_2' + \frac{dg_1}{d\tau_1}\tau_2\tau_1'\right] + \ldots\right\}. \quad (2.26)$$

Substituting these expressions into equations (1.17–(1.22) we obtain the differential equations for the first terms of the series for f_2 and τ_2 and the necessary boundary conditions.

$$\left[g_1(\tau_1 f_2' + \tau_2 f_1')' + \frac{dg_1}{d\tau_1}\tau_2(\tau_1 f_1')'\right]' + (j+1)f_1(\tau_1 f_2' + \tau_2 f_1')' + (j+1)f_2(\tau_1 f_1')' - 2\tau_1 f_1' f_2' - \tau_2 f_1'^2 = 2\beta_1 \quad (2.27a)$$

$$(g_1\tau_2)'' + (1+j)\sigma(f_1\tau_2' + f_2\tau_1') = 0 \quad (2.27b)$$

$$f_2(0) = 0, f_2'(0) = 0, \tau_2(0) = 0 \quad (2.27c)$$

$$f_2''(\infty) = -1, \tau_2(\infty) = 0 \quad (2.27d)$$

with

$$\beta_1 = \lim_{\eta \to \infty} (\eta - f_1). \qquad (2.27e)$$

The $2\beta_1$ term on the right-hand side of equation (2.27a) comes from the second-order pressure gradient due to vorticity interaction. In its original form (before specialization to the stagnation point region) it appears on the right-hand side of the second-order momentum equation (1.18). The second-order pressure gradient is given as follows from equation (1.24) with $j = 1$.

$$p_{2s} = r\,(R_1^2\, T_1\, S_1'\, V_2)_{n=0}.$$

From equation (1.25)

$$V_2 = \lim_{N \to \infty} (v_1 - N\, v_{1N}).$$

Near the stagnation point using equations (2.12), (2.16c) and (2.16d) we have

$$V_2\,(s, 0) = 2w_1 \left(\frac{\mu_{10}}{R_{10}\, w_1}\right)^{1/2} \lim_{\eta \to \infty} (\eta - f_1) + \dots \qquad (2.27f)$$

Therefore we get for P_{2s} near the stagnation point

$$P_{2s} = 2s\, T_{10}\, S_1'\,(0)\, R_{10}^2 \left(\frac{w_1\,\mu_{10}}{R_{10}}\right)^{1/2} \lim_{\eta \to \infty} (\eta - f_1) + \dots \qquad (2.27g)$$

Where $\lim_{\eta \to \infty} (\eta - f_1) = \beta_1$ from (2.27e).

We now see how the second-order pressure gradient near the stagnation point depends on β_1. We also see from equation (2.15) its dependence on the displacement thickness.

The solutions of equations (2.27) represent the second-order effect of vorticity near the stagnation point. Further terms can be found, however they will become quite complicated since they will involve the first two terms in the first-order equations and the first two terms in the second-order equations for the vorticity effect.

IV. *Viscosity law for high free stream Mach numbers and temperatures*

The viscosity law will be taken to be Sutherland's, and will reduce to a simpler law for high free stream Mach numbers and temperatures. This law can be written as follows:

$$\frac{\mu^*}{\mu_r^*} = \frac{t_r^* + C^*}{t^* + C^*} \left(\frac{t^*}{t_r^*}\right)^{3/2}. \qquad (2.28)$$

Where μ_r^* and t_r^* are reference viscosities and temperatures respectively. C^* is a constant which when taken to be 198·6°R gives a good fit of experimental data to the Sutherland law. Using the relation (1.11) we non-dimensionalize the viscosity law so that

$$\mu = \frac{1 + C'}{t + C'}\,(t)^{3/2} \qquad (2.29)$$

where

$$C' = \frac{C_p^* C^*}{U_\infty^{*2}}. \qquad (2.30)$$

From equation (2.7) we have for the first-order temperature the expansion

$$t_1\,(s, N) = T_{10}\,[\tau_1\,(\eta) + \pi_2\tau_3\,(\eta)s^2 + \dots].$$

Therefore substituting this into equation (2.29) and expanding in powers of s we find for the first-order viscosity term

$$\mu(t_1) = (T_{10}\,\tau_1)^{3/2}\,(1 + C') \left\{\frac{1}{T_{10}\,\tau_1 + C'} + \frac{\pi_2\,\tau_3}{2\tau_1}\left[\frac{T_{10}\,\tau_1 + 3C'}{(T_{10}\,\tau_1 + C')^2}\right] s^2 + \dots\right\}. \qquad (2.31)$$

Therefore the coefficients in equation (2.8)

$$\mu(t_1) = \mu_{10}\,[g_1(\eta) + \pi_2\, g_3(\eta)s^2 + \dots]$$

are determined with

$$\mu_{10} = (T_{10})^{3/2}\,\frac{1 + C'}{T_{10} + C'}. \qquad (2.32)$$

We find that

$$g_1 = (\tau_1)^{3/2}\,\frac{T_{10} + C'}{T_{10}\,\tau_1 + C'} \qquad (2.33a)$$

$$g_3 = (\tau_1)^{1/2}\,\tau_3\,\frac{(T_{10} + C')\,(T_{10}\,\tau_1 + 3C')}{2(T_{10}\,\tau_1 + C')^2} \qquad (2.33b)$$

when Sutherland's law is used. With the abbreviation

$$C = \frac{C'}{T_{10}} = \frac{C^*}{\dfrac{U_\infty^{*2}\, T_{10}}{C_p^*}} = \frac{C^*}{[1T_\infty^* + \frac{1}{2}\,(\gamma - 1)\, M_\infty^2]} \qquad (2.34a)$$

we get

$$g_1 = (\tau_1)^{3/2} \frac{1 + C}{\tau_1 + C} \quad (2.34b)$$

$$g_3 = (\tau_1)^{1/2} \tau_3 \frac{(1 + C)(\tau_1 + 3C)}{2(\tau_1 + C)^2}. \quad (2.34c)$$

From equations (2.34b and c) along with (2.34a) we see that when the denominator is sufficiently large (i.e. large free stream Mach number and temperature) C can be neglected in comparison to τ_1 and 1. For this case we get

$$g_1 = (\tau_1)^{1/2}, \quad \frac{dg_1}{d\tau_1} = \frac{1}{2}(\tau_1)^{-1/2}, \quad g_3 = \frac{\tau_3}{2}(\tau_1)^{-1/2}. \quad (2.35a, b, c)$$

We will use this viscosity law in the integration of equations (2.16), (2.17), and (2.27). We could use the more exact Sutherland law, however we would have to integrate the equations every time the free stream conditions are changed (T^*_∞ and M_∞) due to the presence of the constant C. However with equations (2.35) we need to know only $b_0 = T_b(0)/T_{10}$, the ratio of temperatures across the boundary layer at the stagnation point, in order to integrate the equations.

3. INTEGRATION OF THE ORDINARY DIFFERENTIAL EQUATIONS RESULTING FROM THE BLASIUS SERIES

The ordinary differential equations (2.16) for the first term of the first-order boundary layer equations are non-linear, and due to their complexity must be integrated numerically. Modern digital computing machines make this problem simpler even though they do not eliminate all of the problems associated with integrating such equations. After the solutions are obtained for these equations, the solution of the equations for the higher order terms will be simpler since the equations for these terms are linear. The difficulty with the non-linear differential equations for the first terms arise because of the boundary conditions. The two non-linear equations present a system of fifth-order and consequently need five boundary conditions. Three of these conditions are given at $\eta = 0$. The remaining two conditions are given at $\eta = \infty$. This is known as the two-point boundary value problem. The procedure used for the integration was the fourth-order Adams predictor corrector method with accuracy to approximately five decimal places. The boundary conditions at infinity were satisfied by guessing two conditions at $\eta = 0$ and integrating to $\eta = 5$ and seeing if the conditions at infinity are met. The value of $\eta = 5$ was used since it was found that it was sufficiently large for checking that the unknown functions approached their correct values at infinity. The initial estimates were obtained from Table 1 of Cohen and Reshotko [10], where a similar problem was solved using a linear viscosity law. Newton's method as given in Appendix B of Reshotko and Beckwith [11] was then used to compute new starting values and repeat the integration. It was found that after a few trials the procedure converged to the correct boundary conditions. In integrating the second terms for the Blasius series for the first-order boundary-layer equations, universal functions were obtained which depended only upon the ratio b_0 of wall to stagnation-point temperature at the stagnation point. This requires the solution of four universal problems as will be seen later [see (3.5a)], and the superposition of these four problems gives the complete solution for the second terms. The equations for the second-order effect of vorticity at the stagnation point were also solved and required only one solution since they depended only upon the ratio of wall to stagnation-point temperature.

I. *Variables used for the computation of the boundary-layer quantities*

In integrating the equations on the computer it is convenient that the equations be expressed as a set of first-order differential equations. The set of two differential equations is therefore reduced to a set of five first-order equations as follows:

First-order boundary-layer quantities (*see Section* 2, *II*). Let

$$X_1 = f_1, \quad X_2 = f_1' \tau_1 \quad (3.1a, b)$$

$$X_3 = g_1(f_1' \tau_1)', \quad X_4 = \tau_1, \quad X_5 = g_1 \tau_1'. \quad (3.1c, d, e)$$

Therefore simplifying and using equations (2.16) we can get five first-order differential equations for the first terms of the Blasius Series for the first-order boundary-layer equations with $j = 1$ (axisymmetric flow). These equations will not be given here since they can easily be obtained by substitution.

The corresponding boundary conditions are:

at $\eta = 0$

$$X_1(0) = X_2(0) = 0,\ X_4(0) = b_o \quad (3.2a)$$

at $\eta = \infty$

$$X_2(\infty) = X_4(\infty) = 1. \quad (3.2b)$$

For integrating equations (2.17a and b) we let

$$X_6 = f_3 \quad (3.3a)$$

$$X_7 = f_1' \tau_3 + f_3' \tau_1 + f_1' \tau_1 \left(1 - \frac{jr_3}{\pi_2}\right) \quad (3.3b)$$

$$X_8 = \frac{g_3}{g_1} X_3 + g_1 X_7' \quad (3.3c)$$

$$X_9 = \tau_3 \quad (3.3d)$$

$$X_{10} = g_3 \tau_1' + g_1 \tau_3'. \quad (3.3e)$$

Using equations (2.17) and simplifying we can get the equations for the second terms of the Blasius Series for the first-order boundary-layer equations with $j = 1$. As in the case for the first terms these equations will not be given since they can also be obtained by direct substitution.

The corresponding boundary conditions are:

At $\eta = 0$

$$X_6(0) = X_7(0) = 0,\ X_9(0) = \frac{b_2}{\pi_2}. \quad (3.4a)$$

At $\eta = \infty$

$$\left.\begin{aligned} X_7(\infty) &= \frac{1}{4\gamma} + \frac{\pi_4}{2\pi_2^2} \\ X_9(\infty) &= -\left(\frac{\gamma - 1}{\gamma}\right). \end{aligned}\right\} \quad (3.4b)$$

It is important to bear in mind that the equations for X_6 to X_{10} are linear in the dependent variables and that therefore superposition will be permissible. This fact will facilitate the solution for the given boundary conditions.

In the differential equations for the variables and the boundary conditions (X_6 to X_{10}) there are five parameters involved. If we assume that $\gamma = 1{\cdot}4$ and $\sigma = 0{\cdot}7$, and we are only interested in this case, then the number of parameters reduces to three which are b_2/π_2, π_4/π_2^2 and $1 - r_3/\pi_2$. Therefore, to introduce universal functions which when solved determine the second terms once and for all for a given b_o, we let X_i where $i = 6, 7, 8, 9, 10$ be the following:

$$X_i = X_{i1} + \frac{b_2}{\pi_2} X_{i2} + \frac{\pi_4}{\pi_2^2} X_{i3} + \left(1 - \frac{r_3}{\pi_2}\right) X_{i4} \quad (3.5a)$$

with boundary conditions.

At $\eta = 0$

$$X_{61}(0) = X_{62}(0) = X_{63}(0) = X_{64}(0) = 0 \quad (3.5b)$$

$$X_{71}(0) = X_{72}(0) = X_{73}(0) = X_{74}(0) = 0 \quad (3.5c)$$

$$X_{91}(0) = 0,\ X_{92}(0) = 1,\ X_{93}(0) = 0,\ X_{94}(0) = 0. \quad (3.5d)$$

At $\eta = \infty$

$$X_{71}(\infty) = 1/4\gamma,\ X_{72}(\infty) = 0,\ X_{73}(\infty) = 1/2,\ X_{74}(\infty) = 0 \quad (3.5e)$$

$$X_{91}(\infty) = -\left(\frac{\gamma - 1}{\gamma}\right),\ X_{92}(\infty) = 0,\ X_{93}(\infty) = 0,\ X_{94}(\infty) = 0. \quad (3.5f)$$

The differential equations for the universal functions will not be given here, however, they are very easily obtained from the governing equations by neglecting certain terms. To obtain the differential equations for the X_{i1} terms neglect the terms which have b_2/π_2, π_4/π_2^2 and $1 - r_3/\pi_2$ as coefficients. The equations for X_{i2} are obtained by neglecting all terms with subscripts between 1 and 5 only. The equations for X_{i3} are obtained by neglecting terms with subscripts between 1 and 5 only which do not have π_4/π_2^2 as a coefficient, and the equations for X_{i4} are obtained in a like manner by neglecting terms with subscripts between 1 and 5 only which do not have $1 - r_3/\pi_2$ as a coefficient. In each case the terms which are kept do not keep the coefficient (i.e. for X_{i3}, π_4/π_2^2) since the form assumed for the universal functions (equation 3.5a) takes care of this. All of the above is possible since the equations for the variables X_6–X_{10} are linear in the unknown dependent variables.

Given below are the relations for the temperature, velocities, etc., in terms of the new X_i variables. They are obtained by using equations (2.7, 2.10–2.15) along with equations (3.1) and (3.3).

$$t_1 = T_{10}\,(X_4 + \pi_2 X_9 s^2 + \ldots) \qquad (3.6)$$

$$\rho_1 = R_{10}\left[\frac{1}{X_4} - \frac{\pi_2}{X_4}\left(\frac{X_9}{X_4} + 1\right)s^2 + \ldots\right] \qquad (3.7)$$

$$u_1 = w_1\,(X_2 s + \pi_2 X_7 s^3 + \ldots) \qquad (3.8)$$

$$v_1 = -\left(\frac{\mu_{10} w_1}{R_{10}}\right)^{1/2}\left\{2X_1X_4 + \pi_2\left[4X_4X_6 + 2X_1X_4\left(\frac{X_9}{X_4} + 1 - \frac{r_3}{\pi_2}\right)\right]s^2 + \ldots\right\} \qquad (3.9)$$

$$\bar{\tau}_1(0) = (2w_1\pi_2 p_{10}\mu_{10})^{1/2}\,[X_3(0)s + \pi_2 X_8(0)s^3 + \ldots] \qquad (3.10)$$

$$q_1(0) = -\frac{T_{10}}{\sigma}(w_1 R_{10}\mu_{10})^{1/2}\,[X_5(0) + \pi_2 X_{10}(0)s^2 + \ldots] \qquad (3.11)$$

$$\delta^* = \epsilon\left(\frac{\mu_{10}}{w_1 R_{10}}\right)^{1/2}\lim_{\eta\to\infty}\left\{\eta - X_1 - \pi_2\left[X_6 - \left(\frac{\pi_4}{2\pi_2^2} - \frac{3}{4\gamma} + \frac{r_3}{\pi_2}\right)X_1\right]s^2 + \ldots\right\}. \qquad (3.12)$$

All of the preceding relations are in terms of X_1, X_2–X_{10} except where the functions g_1 and g_3 appear which come from the viscosity law. The actual law used in integrating the equations was the square root viscosity law given in equations (2.35). In terms of the X_i's they can be expressed as follows:

$$g_1 = (X_4)^{1/2}, \quad g_3 = \frac{1}{2}\frac{X_9}{(X_4)^{1/2}}. \qquad (3.13, 3.14)$$

Figures 2–14 show the results of integrating the equations for the first and second terms of the Blasius Series for a variety of wall conditions. These figures were computed using the ratio of specific heats $\gamma = 1{\cdot}4$ and Prandtl number $\sigma = 0{\cdot}7$.

Values for the unknown X functions [see (3.1a–c), (3.3a–c) and (3.5a)] are given at $\eta = 0$ and 0·2 in Table 1. These values are helpful for repeating the integration or interpolating to obtain initial conditions to solutions having different values of b_o (the ratio of wall to stagnation point temperature) from those considered herein.

Figures 2–14 show some of the functions or their boundary values X_{ij} in diagrams: $i = 1, \ldots 9$, $j = 1, \ldots 4$. Figs. 2–6 show the quantities which determine the coefficients in the u_1 component of velocity, and the t_1 temperature expression of equations (3.8) and (3.6). Equation (3.5a) must be used for superimposing the universal functions to get the second terms in these expressions. We also see that with the use of Figs. 7–11 we can compute the first-order shear stress and heat transfer at the wall by using equations (3.10) and (3.11). Finally with the help of Figs. 12–14 and equation (3.12) we can compute the displacement thickness.

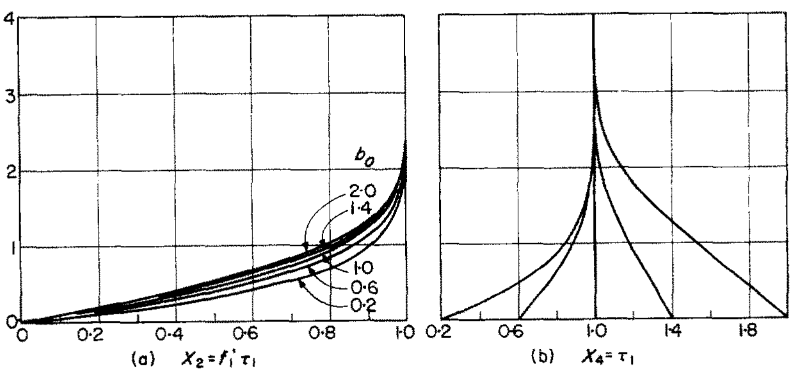
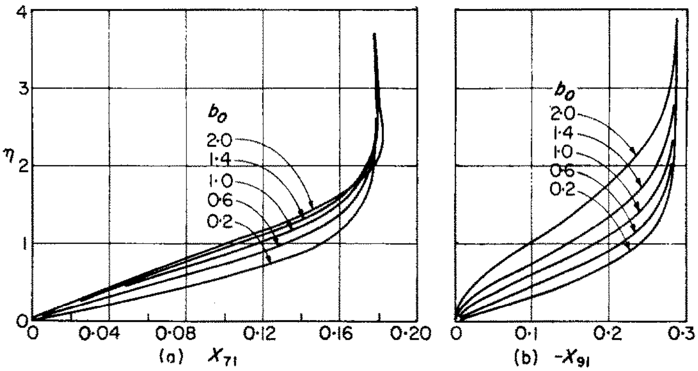


FIG. 2. Coefficients used in the first terms of equations (3.8) and (3.6).



FIGS. 3–6. Coefficients used in the second terms of equations (3.8) and (3.6) with equation (3.5a).

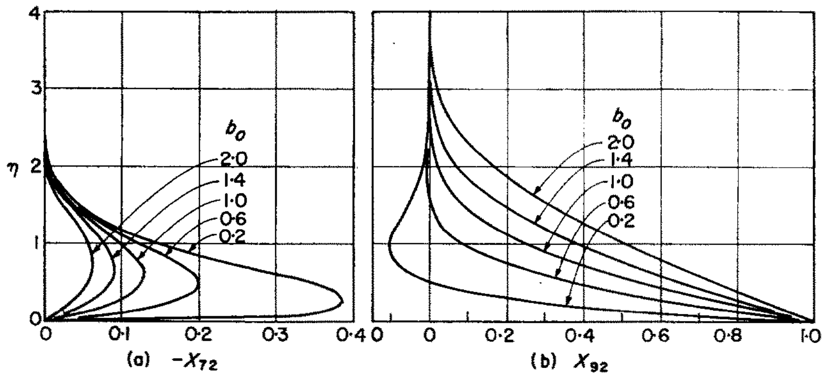


FIG. 4.

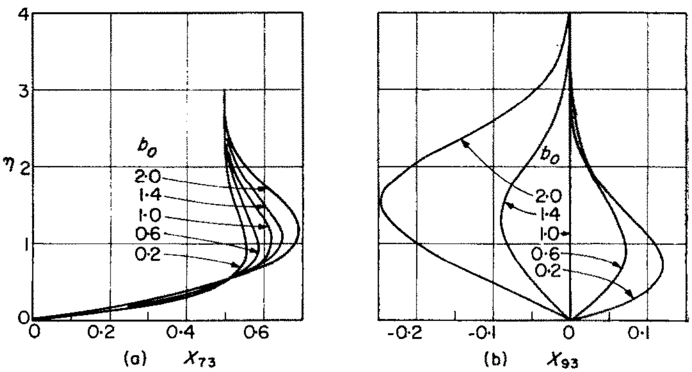


FIG. 5.

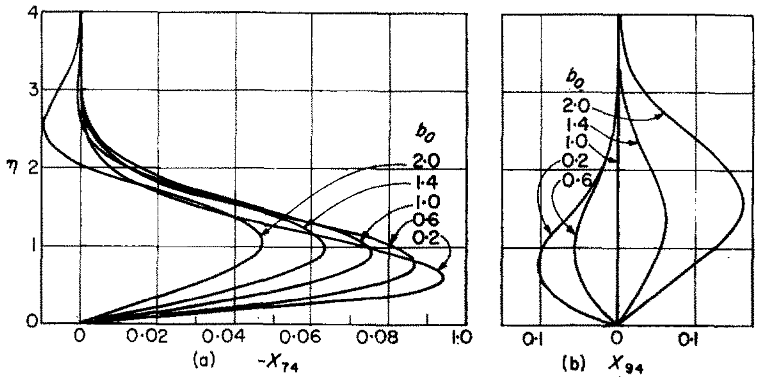


FIG. 6.

Table 1

(a) *First-order results—First terms*

b_0	η	X_1	X_2	X_3	X_4	X_5
0·2	0·000	0·000000	0·000000	1·093079	0·200000	0·578725
	0·200	0·118469	0·363445	0·892209	0·409440	0·566372
0·6	0·000	0·000000	0·000000	1·206979	0·600000	0·274214
	0·200	0·044800	0·278235	1·007253	0·668769	0·272744
1·0	0·000	0·000000	0·000000	1·311938	1·000000	0·000000
	0·200	0·024906	0·242394	1·112107	1·000000	0·000000
1·4	0·000	0·000000	0·000000	1·408494	1·400000	−0·260791
	0·200	0·016635	0·222912	1·208600	1·355579	−0·260443
2·0	0·000	0·000000	0·000000	1·540999	2·000000	−0·638093
	0·200	0·010832	0·206105	1·341060	1·908727	−0·637628

(b) *First-order results—Second terms*

b_0	η	X_{61}	X_{71}	X_{81}	X_{91}	X_{101}
0·2	0·000	0·000000	0·000000	0·057596	0·000000	−0·129477
	0·200	0·023115	0·036585	0·058965	−0·063767	−0·253177
0·6	0·000	0·000000	0·000000	0·077651	0·000000	−0·084935
	0·200	0·004859	0·022805	0·077976	−0·035925	−0·196271
1·0	0·000	0·000000	0·000000	0·094523	0·000000	−0·044255
	0·200	0·002247	0·019911	0·094697	−0·020607	−0·152178
1·4	0·000	0·000000	0·000000	0·109348	0·000000	−0·005545
	0·200	0·001438	0·018947	0·109470	−0·010959	−0·113604
2·0	0·000	0·000000	0·000000	0·128976	0·000000	0·050291
	0·200	0·000943	0·018419	0·129062	−0·001298	−0·060616

b_0	η	X_{62}	X_{72}	X_{82}	X_{92}	X_{102}
0·2	0·000	0·000000	0·000000	0·109830	1·000000	−1·208579
	0·200	−0·370879	−0·382648	0·165144	0·346686	−0·991755
0·6	0·000	0·000000	0·000000	0·161098	1·000000	−1·012232
	0·200	−0·082071	−0·147418	0·167336	0·704916	−0·950033
1·0	0·000	0·000000	0·000000	0·160551	1·000000	−0·942577
	0·200	−0·030297	−0·077936	0·162536	0·813731	−0·909857
1·4	0·000	0·000000	0·000000	0·154131	1·000000	−0·904881
	0·200	−0·014842	−0·049086	0·155070	0·862082	−0·883284
2·0	0·000	0·000000	0·000000	0·143695	1·000000	−0·871687
	0·200	−0·006807	−0·029490	0·144126	0·898135	−0·857623

Table 1—continued

b_o	η	X_{63}	X_{73}	X_{83}	X_{93}	X_{103}
0·2	0·000	0·000000	0·000000	1·106499	0·000000	0·217023
	0·200	0·093869	0·317970	0·733137	0·066260	0·200121
0·6	0·000	0·000000	0·000000	1·310664	0·000000	0·107443
	0·200	0·044840	0·277228	0·920371	0·026119	0·104979
1·0	0·000	0·000000	0·000000	1·481382	0·000000	0·000000
	0·200	0·026971	0·256529	1·086410	0·000000	0·000000
1·4	0·000	0·000000	0·000000	1·631847	0·000000	−0·107328
	0·200	0·018762	0·244745	1·235033	−0·018408	−0·106677
2·0	0·000	0·000000	0·000000	1·832201	0·000000	−0·268678
	0·200	0·012659	0·234477	1·434165	−0·038862	−0·267779

b_o	η	X_{64}	X_{74}	X_{84}	X_{94}	X_{104}
0·2	0·000	0·000000	0·000000	−0·207889	0·000000	−0·173619
	0·200	−0·126200	−0·058327	−0·184777	−0·053207	−0·162112
0·6	0·000	0·000000	0·000000	−0·158451	0·000000	−0·078575
	0·200	−0·049893	−0·037315	−0·151607	−0·019127	−0·077154
1·0	0·000	0·000000	0·000000	−0·126834	0·000000	0·000000
	0·200	−0·027435	−0·025187	−0·123384	0·000000	0·000000
1·4	0·000	0·000000	0·000000	−0·103018	0·000000	0·070612
	0·200	−0·018031	−0·017936	−0·100830	0·012116	0·070299
2·0	0·000	0·000000	0·000000	−0·075244	0·000000	0·167910
	0·200	−0·011496	−0·011343	−0·073881	0·024293	0·167511

(c) *Second-order results*

b_o	η	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}
0·2	0·000	0·000000	0·000000	−0·877865	0·000000	−0·212018
	0·200	−0·081106	−0·300163	−0·799847	−0·065140	−0·199781
0·6	0·000	0·000000	0·000000	−1·303454	0·000000	−0·131161
	0·200	−0·047687	−0·308108	−1·152270	−0·031935	−0·128920
1·0	0·000	0·000000	0·000000	−1·768602	0·000000	0·000000
	0·200	−0·033856	−0·330981	−1·541463	0·000000	0·000000
1·4	0·000	0·000000	0·000000	−2·260686	0·000000	0·173065
	0·200	−0·027022	−0·359451	−1·958985	0·029695	0·172268
2·0	0·000	0·000000	0·000000	−3·035639	0·000000	0·501336
	0·200	−0·021578	−0·405141	−2·629140	0·072528	0·500042

When b_2, π_2, r_3, etc., are known in a specific case and high accuracy in the temperature, velocity, etc., in the boundary layer are required, we can compute the exact initial conditions X_3, X_5, X_8 and X_{10} with equation (3.5a) and the use of the initial conditions on the universal functions. We then use the governing equations to integrate directly to find the required quantities. In doing this we are, however, limited to the specific cases of wall conditions computed as examples.

Second-order boundary-layer quantities (see section 2, III). For the second-order effect of vorticity we define X_{11} through X_{15} as follows:

$$X_{11} = f_2, \quad X_{12} = \tau_1 f_2' + \tau_2 f_1' \qquad (3.15a, b)$$

$$X_{13} = g_1 X_{12}', \quad X_{14} = \tau_2 \qquad (3.15c, d)$$

$$X_{15} = g_1 \tau_2' + \frac{dg_1}{d\tau_1} \tau_2 \tau_1'. \qquad (3.15e)$$

Equations (2.27a, b) yield five first-order differential equations for the variables X_{11} to X_{15}. These equations will not be given here since they can be obtained by direct substitution. The corresponding boundary conditions are [see (2.27c, d, e)]:

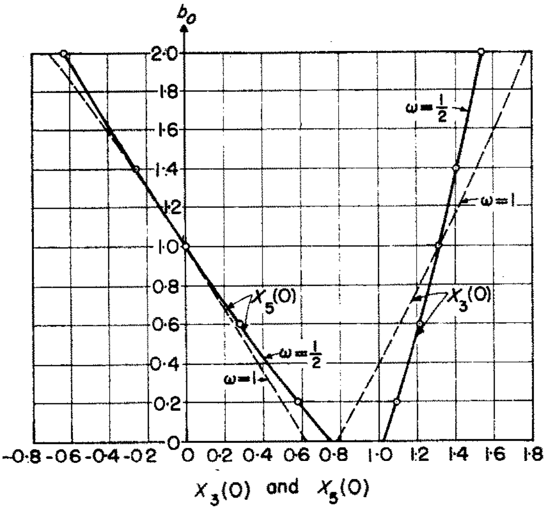


FIG. 7. Coefficients used in the first terms of equations (3.10) and (3.11).

At $\eta = 0$

$$X_{11}(0) = X_{12}(0) = X_{14}(0) = 0. \quad (3.16a)$$

At $\eta = \infty$

$$\left.\begin{aligned} X_{13}(\infty) &= -1\dagger \\ X_{14}(\infty) &= 0. \end{aligned}\right\} \quad (3.16b)$$

The relations for the second-order temperature, velocities, etc., in terms of the X_i variables are:

$$t_2 = \frac{T_{10}^2}{w_1} S_1'(0) \left(\frac{R_{10}\mu_{10}}{w_1}\right)^{1/2} [X_{14} + \dots] \quad (3.17)$$

$$\rho_2 = -\frac{R_{10}T_{10}S_1'(0)}{w_1} \left(\frac{R_{10}\mu_{10}}{w_1}\right)^{1/2} \left(\frac{X_{14}}{X_4^2}\right) + \dots \quad (3.18)$$

$$u_2 = T_{10} S_1'(0) \left(\frac{R_{10}\mu_{10}}{w_1}\right)^{1/2} [X_{12}s + \dots] \quad (3.19)$$

$$v_2 = -2\frac{T_{10}\mu_{10}S_1'(0)}{w_1} [(X_4X_{11} + X_1X_{14}) + \dots] \quad (3.20)$$

$$\bar{\tau}_2(0) = \mu_{10}R_{10}T_{10}S_1'(0)\,[X_{13}(0)s + \dots] \quad (3.21)$$

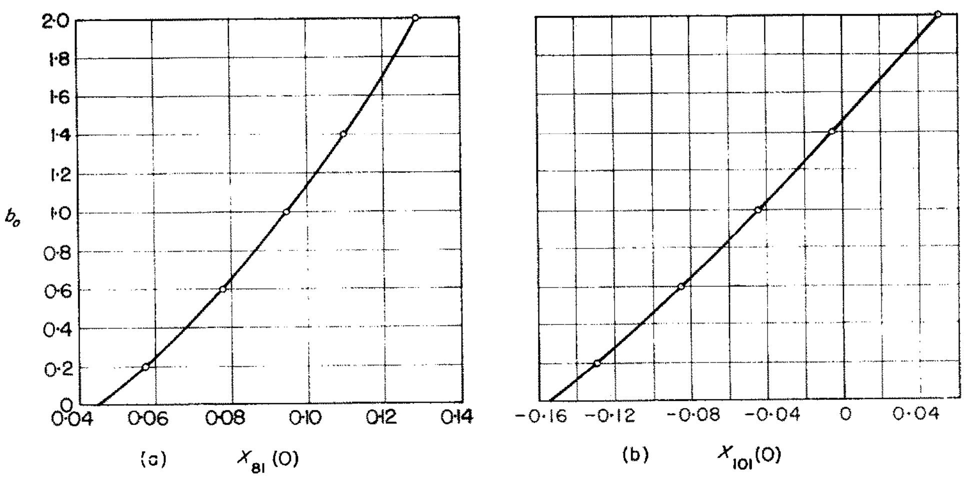
$$q_2(0) = -\frac{1}{\sigma}\frac{R_{10}\mu_{10}}{w_1} T_{10}^2 S_1'(0)\,[X_{15}(0) + \dots]. \quad (3.22)$$

In integrating the equations for X_{11}–X_{15} the relations for g_1 and g_3 for the square root viscosity law were used as previously given.

† The condition at infinity on X_{13} given by equation (3.16g) can be explained as follows.
Using equations (3.15c) and (3.15b) we get

$$X_{13} = g_1(\tau_1 f'_2 + \tau_1 f')'.$$

Knowing the asymptotic nature of the above functions as η goes to infinity we obtain $\tau_1 f'_2 \sim (-\eta)$ from (2.16d) and (2.27d), $\tau_2 f'_1 \sim 0$ from (2.27d) and (2.16d), and $g_1 \sim 1$ from (2.34a) and (2.16d). Substituting this into the expression for X_{13} we get $X_{13} \sim 1(-\eta)'$ or $X_{13} \sim -1$ as $\eta \to \infty$.



FIGS. 8–11. Coefficients used in the second terms of equations (3.10) and (3.11) with equation (3.5a).

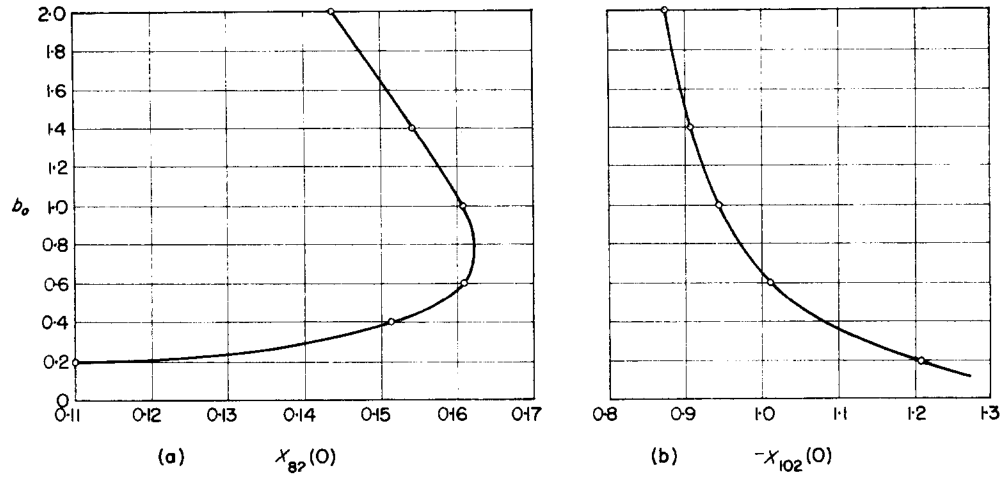


FIG. 9.

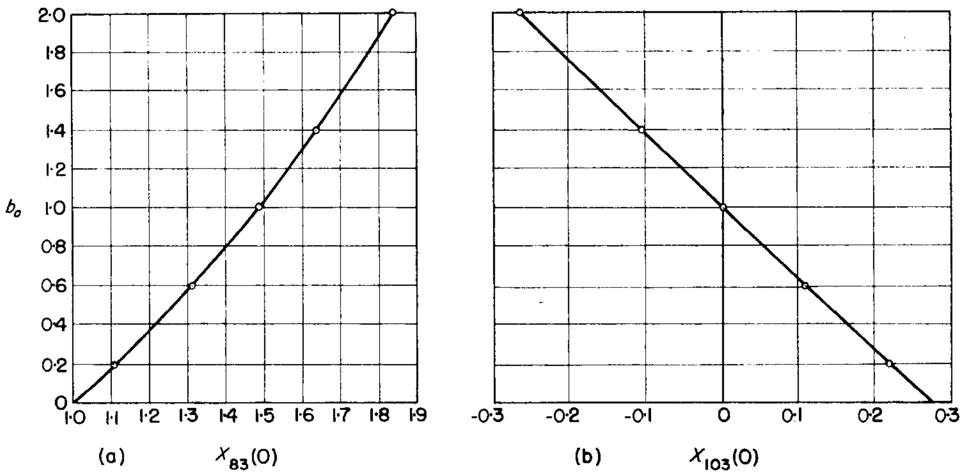


FIG. 10.

In addition the following is the proper definition for $dg_1/d\tau_1$ as given by equation (2.35b).

$$\frac{dg_1}{d\tau_1} = \frac{1}{2}\,\frac{1}{(X_4)^{1/2}}. \tag{3.23}$$

As in the first-order case, results for the second-order terms are given in diagrams; see Figs. 15 and 16. Fig. 15(a) shows a plot of X_{12} vs. η which when used with equation (3.19) determines the second-order contribution to the velocity profile parallel to the wall. Similarly Fig. 15(b) shows a plot of X_{14} which when used with equation (3.17) determines the second-order contribution to the temperature profile. Figs. 16(a) and 16(b) along with equations (3.21) and (3.22) determine the second-order contributions to shear stress and heat transfer at the wall. A square root viscosity law was used for computing most of the points given on these figures, however, two points are included for the linear viscosity law so that the influence of a different viscosity law can be seen. As in the first-order case values for the unknown X functions (3.15a–c) are given at $\eta = 0$ and 0·2 in Table 1.

Finally by using equations (1.3), (1.26), (1.27) and (1.31) we can find the total flow quantities

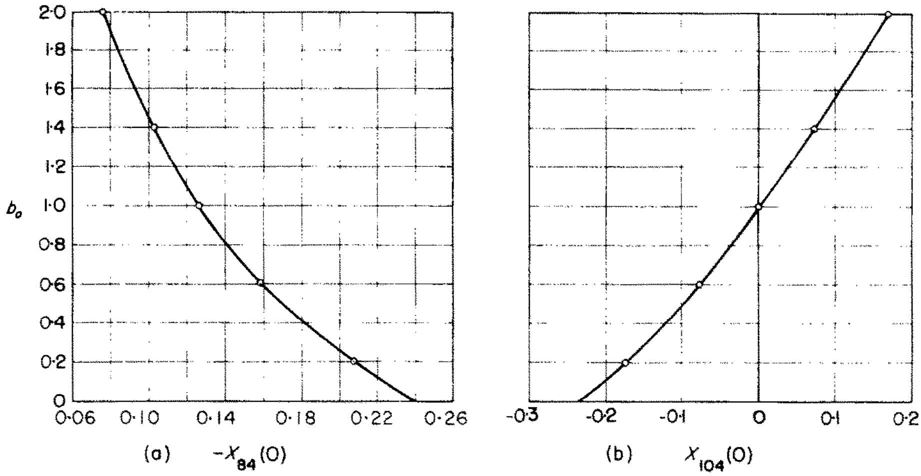


FIG. 11.

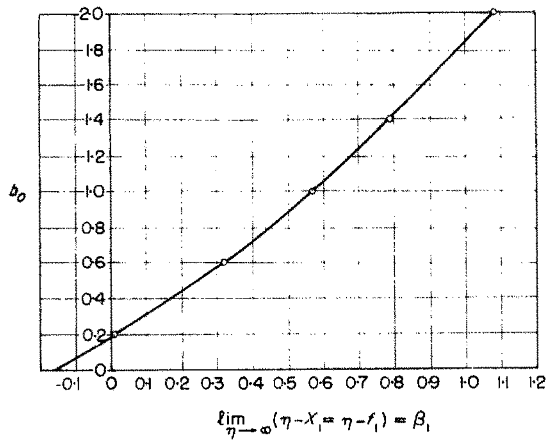


FIG. 12. Coefficient used in the first term of equation (3.12).

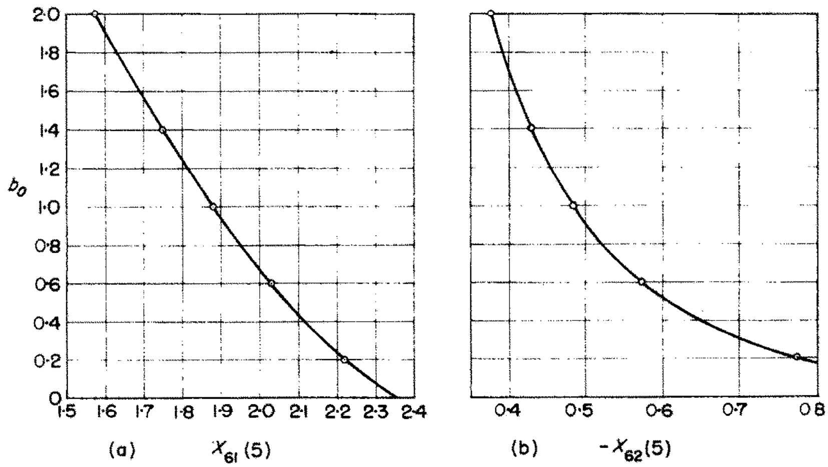


FIG. 13. Coefficients used in the second term of equation (3.5a).

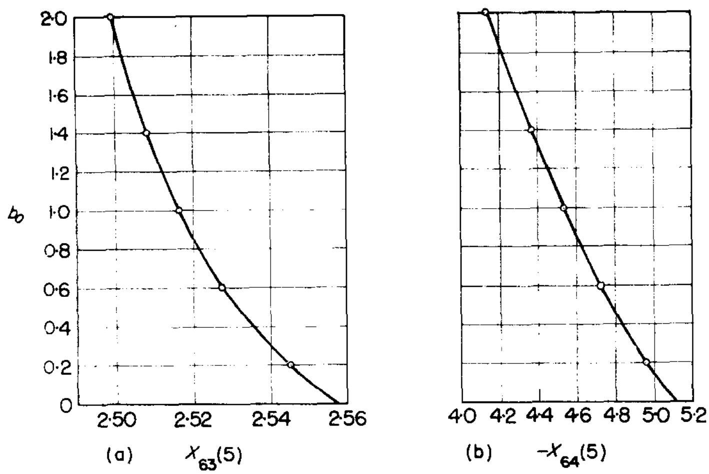


FIG. 14. Coefficients used in the second term of equation (3.5a).

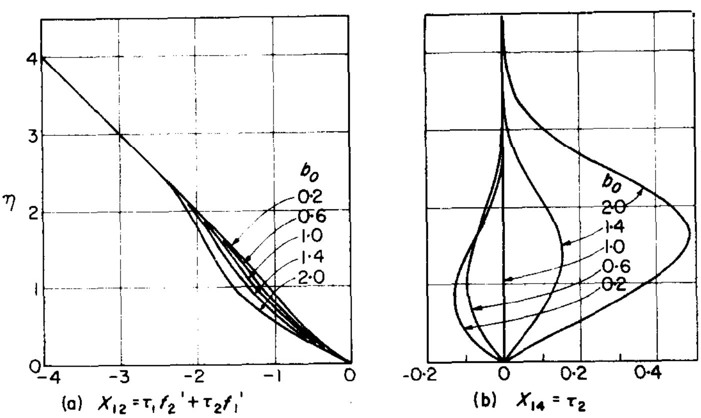


FIG. 15. Coefficients used in equations (3.19) and (3.17).

in the boundary layer including first- and second-order effects.

4. COMPARISON WITH OTHER THEORIES AND EXPERIMENTS

There have been several methods developed for treating the problem of viscous compressible flow over blunt axisymmetric bodies in the Reynolds-number range where Reynolds numbers are too small for the conventional first-order boundary-layer equations to give reasonable results. These methods fall into essentially two categories. In the first category the idea of a boundary-layer with an external inviscid flow region is used. In the second category the entire flow field is treated at once including the necessary viscous effects. There are several methods which fall under the first category. One of these is to approach the problem as a singular perturbation problem and to use the method of inner and outer expansions to find the first- and second-order boundary-layer equations. Examples of this method are the methods of Van Dyke [1], which is followed in this paper, Lenard [19], and Maslen [3, 18]. Another method which falls in the first category is the method where the conventional boundary-layer equations are modified to take care of vorticity or other higher order effects. In this method the idea of the boundary-layer accompanied by an outer inviscid

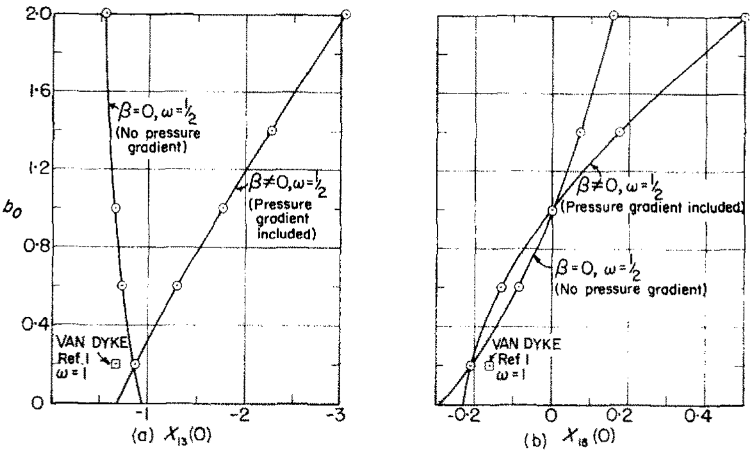


FIG. 16. Coefficients used in equations (3.21) and (3.22).

region is retained, but the conventional first-order boundary-layer equations are modified to take care of the higher order effects. An example of this method is the method of Ferri, Zakkay and Ting [5]. The second category consists of retaining the essential terms in the Navier–Stokes equations to obtain a viscous flow model which reflects the essential effects which are under study. One example of this is the method of Cheng [4] or Cheng and Chang [17] where the thin shock layer approximation is applied to the Navier–Stokes equations to describe the flow around a blunt axisymmetric body, and solutions in the stagnation region are obtained. Another of this type of solution is that of Ho and Probstein [13].

In order to verify any theory experiments are necessary, and unfortunately the experiments in verifying the various second-order theories are quite difficult and definite answers have not as yet been obtained. The experiments carried out thus far, like the theory, do not show the agreement that one would hope for. One group of experiments, carried out by Ferri, Zakkay and Ting [5] in the hypersonic wind tunnel at the Polytechnic Institute of Brooklyn agree very well with their own theory. Another set of experiments carried out in the low density wind tunnel at the University of California by Hickman [14], on the other hand, agree well with Van Dyke's theory but cannot be used to rule out the results of the authors who neglected the second-order pressure gradient in the vorticity term (i.e. Maslen, and Hayes and Probstein). The scatter in Hickman's data is sufficient so that the cases of second-order pressure gradient included and also no second-order pressure gradient fall within the scatter. If Hickman's experiments are correct, they do, however, tend to rule out the theory of Ferri, Zakkay and Ting since most of the experimental data of Hickman falls below Ferri, Zakkay and Ting's theory and experiments. A third set of experiments are the shock tube experiments run at the Cornell Aeronautical Lab. by Wilson and Wittliff [15]. They tend to show an increase in heat transfer over the first-order boundary-layer as all of the other experiments have shown but there is enough spread in their data so that it cannot be used to rule out any of the theories discussed herein. All of the theories discussed herein show an increase in heat transfer due to the effect of external vorticity and fall within Wilson and Wittliff's data. All of this indicates that while experiments have been run, no definite conclusions can be drawn as yet as to the most correct ones. A great deal of experimental work must still be done before any of the theories developed thus far can be justified on an experimental basis. Another point which has often been overlooked is that there are other second-order effects besides the effects of external vorticity. The second-order effects have not been separated out in the experiments so the measured experimental results always include all of these effects. Van Dyke [16] has shown that when all of the effects are included, the agreement between him and Hickman's least squares fit to experimental data is not nearly as good as the agreement when only the vorticity term is included. These other second-order terms have not been included by Ferri *et al.* in their theoretical analysis, so it is questionable as to whether their experimental data should agree with their theory if you assume that their theory is correct.

In order to compare the results obtained herein with other theories and experiments we introduce in the following sections the notations of the other authors referred to previously.

I. *The vorticity interaction parameter*

Hayes and Probstein [2] p. 370 define a vorticity interaction parameter, which is defined as the ratio of the vorticity at the outer edge of the boundary-layer to an average vorticity across the boundary-layer. Taking their definition and using the variables introduced herein we find that in the stagnation-point region, i.e. in keeping only the first term in the Blasius Series in both the first- and second-order theories,

$$\Omega_p - \frac{1}{\sqrt{(2)}} \frac{T_{10} S_1'(0)}{w_1} \left(\frac{R_{10} T_{10}{}^{\omega}}{w_1}\right)^{1/2} \left(\frac{1}{b_o}\right)^{(1-\omega/2)} \epsilon \tag{4.1}$$

where Ω_p is the vorticity interaction parameter, ω is the power in the viscosity law $\mu \alpha T^{\omega}$, and all other symbols are as previously defined. Notice that according to this definition of the vorticity interaction parameter $\Omega_p \rightarrow \infty$ as $b_o \rightarrow 0$ for $\omega < 1$. For our case $\omega = 1/2$, therefore, the interaction parameter becomes infinite as the ratio of wall to stagnation-point temperature goes to zero. This is due to an improper definition of a suitable boundary-layer thickness by Hayes and Probstein, since by using their definition the "suitable" thickness of the boundary-layer goes to ∞ as $b_o \rightarrow 0$. This difficulty does not arise when $\omega = 1$ since the term containing b_o drops out and Ω_p remains bounded as $b_o \rightarrow 0$. Therefore, for a highly cooled body ($b_o \rightarrow 0$) this definition makes sense only for the linear viscosity law. This difficulty arises because of the reference values of ρ^* and μ^* appearing in the transformations given by equations (8.29) of Hayes and Probstein [2]. Instead of using ρ^*_{wall} and μ^*_{wall} in the transformations, values could be used which do not cause trouble as the ratio of wall to stagnation-point temperature, goes to zero. One possible choice would be to use reference values such as Cheng [4] used. He employs a reference temperature in the stagnation-region of $(t^*_{\text{wall}} + t^*_{\text{shock}})/2$. When this value is used in the reference viscosity and density instead of t^*_{wall} then no trouble arises as b_o goes to zero and Ω, the new vorticity interaction parameter, behaves properly.

For the sake of comparison with other results the stagnation-point heat transfer and skin friction can be written as follows using the vorticity interaction parameter. Using the first term of equations (2.13) and (2.14) and equations (2.25) and (2.26) and dividing through by the first-order term we have:

$$\bar{\tau} \; \alpha \; 1 - \sqrt{(2)}\, b_o{}^{(1-\omega/2)} \frac{f_2''(0)}{f_1''(0)} \Omega_p \tag{4.2}$$

$$q \; \alpha \; 1 - \sqrt{(2)}\, b_o{}^{(1-\omega/2)} \frac{\tau_2'(0)}{\tau_1'(0)} \Omega_p \tag{4.3}$$

or from equations (3.10) and (3.11), and (3.21) and (3.22)

$$\bar{\tau} \; \alpha \; 1 - \sqrt{(2)}\, b_o{}^{(1-\omega/2)} \frac{X_{13}(0)}{X_3(0)} \Omega_p \tag{4.4a}$$

$$q \; \alpha \; 1 - \sqrt{(2)}\, b_o{}^{(1-\omega/2)} \frac{X_{15}(0)}{X_5(0)} \Omega_p. \tag{4.5a}$$

Figures 17 and 18 show the results of plotting $-X_{13}(0)/X_3(0)$ and $-X_{15}(0)/X_5(0)$ versus b_o. These are the important quantities in the shear stress and heat-transfer expressions of equations (4.4) and (4.5). The reference to points of other authors is given in a form similar to that of Probstein in a poll of the various authors who have computed the second-order effect of vorticity. These points have been checked against data published since the time of the poll and the results are listed in Table 2 with some additions to include authors not included in Probstein's original poll. Several points given by Probstein have not been included; however, they also fall near the curve given for no second-order pressure gradient [see equation (2.27f)]. The numerical results of Hayes and Probstein

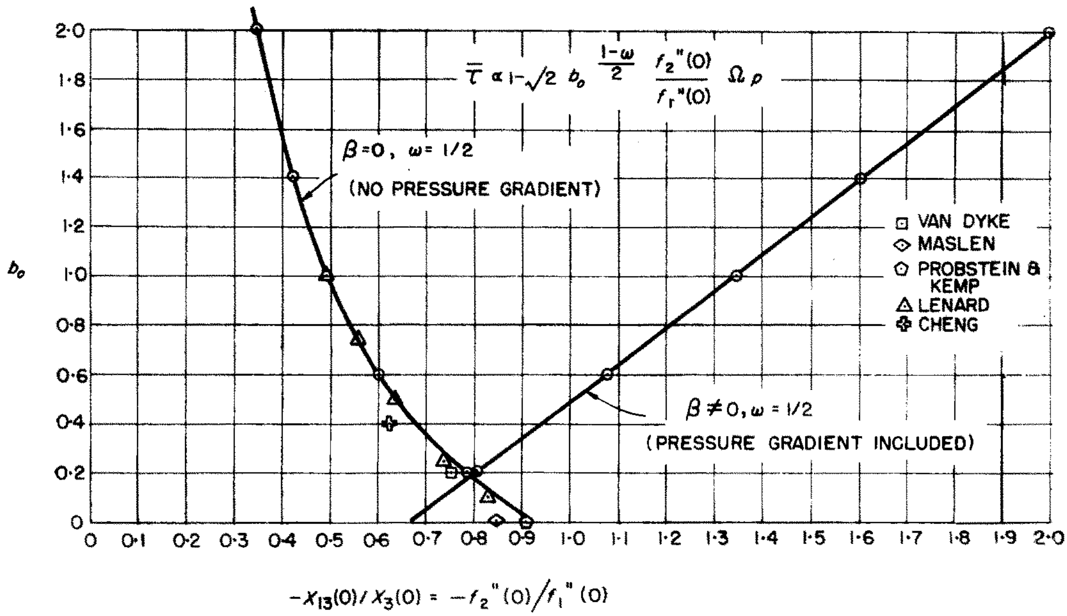


FIG. 17. Coefficient in the skin friction expression in Hayes and Probstein's [2] form [see equations (4.2) and 4.4)].

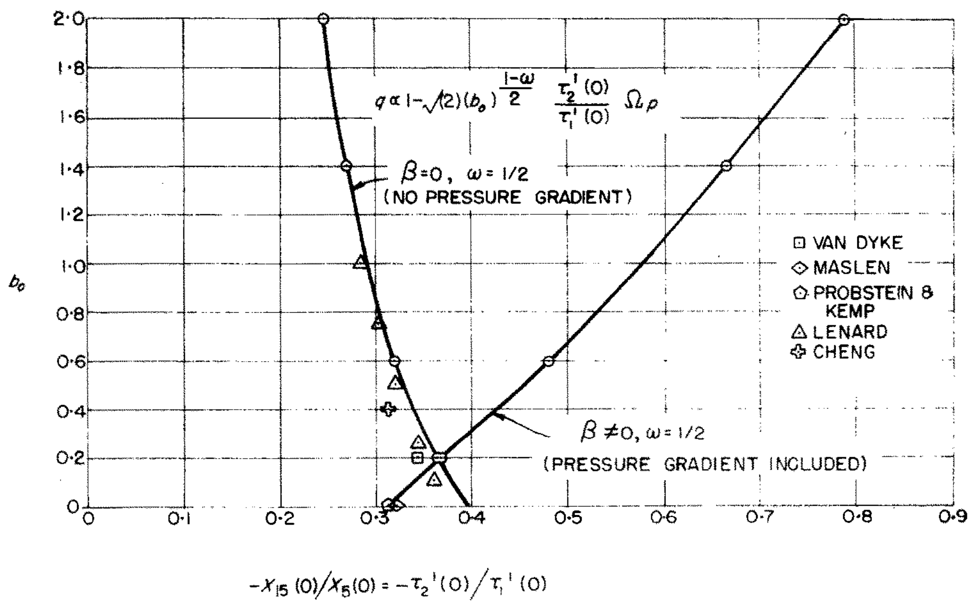


FIG. 18. Coefficient in the heat-transfer expression in Hayes and Probstein's [2] form [see equations (4.3) and (4.5)].

Table 2. Comparison with other authors

(a) *Second-order pressure gradient not included*

Author	a_1	b_1	b_o	σ	ω	Reference
Probstein and Kemp	0·44	1·28	→0	1·0	1	Unpublished
Cheng and Chang	0·495	1·34	→0	0·71	1	17
	0·44	0·88	0·40	0·71	1	17
Maslen	0·45	1·2	→0	1·0	1	18
Lenard	0·403	0·700	1·00	0·76	0·58	19
	0·425	0·783	0·75	↓	↓	↓
	0·451	0·893	0·50			
	0·484	1·044	0·25			
	0·508	1·169	0·10			

(b) *Second-order pressure gradient included*

Author	a_1	b_1	b_o	σ	ω	Reference
Van Dyke	0·482	1·062	0·20	0·70	1	1
Results obtained herein	1·1111	2·7858	2·00	0·70	0·50	
	0·9385	2·2698	1·40	↓	↓	
	0·8150	1·9064	1·00			
	0·6764	1·5272	0·60			
	0·5181	1·1357	0·20			

[2] p. 372, have not been included since, as Cheng and Chang [17] point out, no comparison can be made since some of the conditions under which the computations were made have not been included in Hayes and Probstein's book.

If we let

$$q \propto 1 + a_1 b_o^{(1-\omega/2)}\, \Omega_p \qquad (4.5b)$$

$$\bar{\tau} \propto 1 + b_1 b_o^{(1-\omega/2)}\, \Omega_p \qquad (4.4b)$$

where

$$a_1 = -\sqrt{(2)}\,\frac{X_{15}(0)}{X_5(0)} \qquad (4.5c)$$

$$b_1 = -\sqrt{(2)}\,\frac{X_{13}(0)}{X_3(0)} \qquad (4.4c)$$

then we can compute the quantities given in Table 2.†

† Added in proof: It should be pointed out here that the values given for Lenard [19] in Table 2 and Figures 17 and 18 have been corrected in an addendum to his original work and bring complete agreement with the results obtained herein.

II. *The Reynolds number defined by Ferri, Zakkay and Ting*

In order to compare the results obtained herein with those of Ferri, Zakkay and Ting [5] we must introduce their Reynolds number which is defined as

$$R_f = \frac{\rho_o^* a^* \sqrt{(h_{so}^*)}}{\mu_o^*} \qquad (4.6a)$$

where ρ_o^* = stagnation point density, h_{so}^* = free stream stagnation enthalpy, μ_o^* = viscosity for stagnation conditions.

Using the definitions for ϵ and R_f and also using some thermodynamic relations we can show that

$$\epsilon = \left[\frac{(\gamma-1)M_\infty^2}{1+\frac{\gamma-1}{2}M_\infty^2}\right]^{(2\omega-1/4)} \sqrt{(R_{10})}\,\frac{1}{\sqrt{(R_f)}}. \qquad (4.6b)$$

Figure 19 shows a comparison of results obtained herein with those of Ferri, Zakkay and

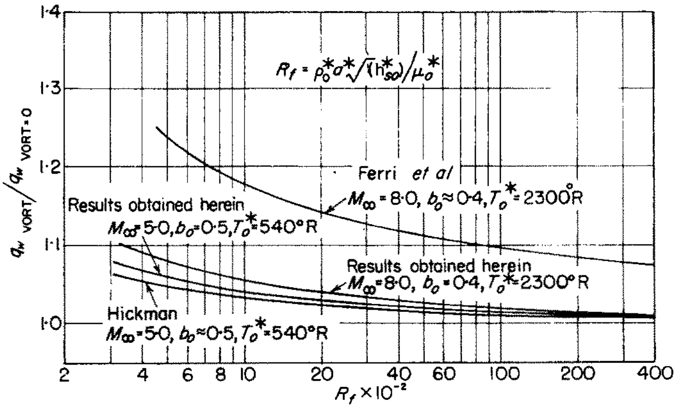


FIG. 19. Comparison of results contained herein with Ferri, Zakkay and Ting [5], and Hickman [14].

Ting [5], Fig. 9. The values used in plotting the lower curve of the figure were obtained from Fig. 18 for $b_0 = 0\cdot4$. We see that the results obtained herein do not agree well with Ferri, Zakkay and Ting.

III. *The Reynolds number of Hickman*

Hickman [14] has defined a Reynolds number as follows:

$$Re_s = \frac{U_s^* \rho_s^* D^*}{\mu_s^*} \tag{4.7a}$$

where U_s^*, ρ_s^*, and μ_s^* are the velocity, density and viscosity respectively behind the shock, and D^* is the body nose diameter.

Using this definition of Re_s, the definition of ϵ, and some thermodynamic relations we find that

$$\epsilon = \sqrt{(2)} \left[\frac{(\gamma - 1)(\gamma + 1)^2 M_\infty^4}{[2\gamma M_\infty^2 - (\gamma - 1)][(\gamma - 1)M_\infty^2 + 2]}\right]^{\omega/2} \frac{1}{\sqrt{(Re_s)}}. \tag{4.7b}$$

If this relation is used along with (4.6) then we can plot Hickman's least squares fit to his experimental data on Fig. 19 for comparison with other theories and experiments. We see that the results of Hickman agree well with the results obtained herein while Ferri *et al.*'s results do not. We should remember, however, that for a proper comparison we should include all of the second-order effects in the analysis since Hickman's experiments include them.

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Résumé—On a examiné l'influence d'un écoulement rotationnel extérieur sur une couche limite laminaire compressible dans la région du point d'arrêt d'un corps arrondi de révolution. Dans le cas considéré le tourbillon est engendré par une onde de choc courbe formée devant le corps qui se déplace à une vitesse supersonique. La méthode de solution utilisée est due à Van Dyke et consiste à résoudre les équations de la couche limite du 1er ordre (ou de Prandtl) et ensuite à résoudre des équations du second ordre pour l'effet du tourbillon en retenant les termes du second ordre par rapport à un paramètre de perturbation relié à l'inverse de la racine carrée d'un nombre de Reynolds.

La première étape de la solution à la fois des équations du 1er ordre et du second ordre pour l'interaction turbulente est la réduction des équations aux dérivées partielles à des équations différentielles au moyen d'un développement en série de Blasius. On a intégré alors numériquement les équations différentielles obtenues. On a présenté les résultats de l'intégration des équations pour les deux premiers termes de la série correspondant aux équations du 1er ordre et pour un terme de la série correspondant aux équations du 2ème ordre. Ces résultats ont été obtenus pour une gamme de rapports de température de paroi à la température du point d'arrêt. Les résultats obtenus à partir de l'intégration de ces équations sont comparés avec les théories et les expériences d'autres auteurs. On a trouvé une bonne concordance avec les théories de certains auteurs lorsqu'on a supposé nul un gradient de pression dû à l'interaction tourbillonnaire. On a trouvé cependant que cette simplification n'est pas permise en général. On a trouvé aussi une concordance avec un groupe de résultats expérimentaux, mais cette comparaison n'a pas beaucoup de sens puisque l'analyse actuelle ne comprend seulement que l'effet du tourbillon et non d'autres effets de second ordre qui peuvent également être importants. Cependant puisque les équations du second ordre sont linéaires les autres effets du second ordre peuvent être calculés séparément et superposés à l'effet tourbillonnaire afin d'obtenir une théorie complète de second ordre.

Zusammenfassung—Der Einfluss äusserer Verwirbelung auf die laminare, kompressible Grenzschicht wird im Staupunktbereich eines achssymmetrischen, stumpfen Körpers untersucht. Im betrachteten Fall wird die Verwirbelung von einer gekrümmten Stosswelle erzeugt, die von dem mit Überschall bewegten Körper hervorgerufen wird. Die Lösungsmethode stammt von Van Dyke und beruht darauf, erst Grenzschichtgleichungen erster Ordnung (nach Prandtl) und dann für den Wirbeleffekt Gleichungen zweiter Ordnung eines Störparameters zu verstehen, der auf die reziproke Quadratwurzel einer Reynoldszahl bezogen wird.

Der erste Schritt in der Lösung, sowohl der Gleichungen erster als auch zweiter Ordnung, für die Wirbelwechselwirkung, besteht darin, die partiellen Differentialgleichungen nach einer Blasius-Reihenentwicklung auf gewöhnliche Differentialgleichungen zu reduzieren. Die resultierenden gewöhnlichen Differentialgleichungen werden numerisch integriert. Integrationsergebnisse sind angegeben für die Gleichung erster Ordnung für die ersten beiden Glieder der Reihe und für ein Glied der Reihe der Gleichungen zweiter Ordnung. Diese Ergebnisse liessen sich für eine Vielzahl von Üerhältnissen der Wandtemperatur zue Staupunkttemperatur erhalten. Die aus die Integration der Gleichungen erzielten Ergebnisse werden mit der Theorie und den Versuchen anderer Autoren verglichen. Gute Übereinstimmung mit anderen Autoren ergibt sich, wenn der auf der Wirbelwechselwirkung beruhende Druckgradient gleich Null gesetzt wird. Allgemein ist diese Vereinfachung—wie sich ergab—jedoch nicht statthaft. Für eine bestimmte Versuchsreihe zeigte sich ebenfalls Übereinstimmung, doch ist ein Vergleich nicht charakteristisch, da die gegenwärtige Analyse nur den Einfluss der Verwirbelung einschliesst und keine anderen Effekte zweiter Ordnung, die gleich wichtig sein können. Da jedoch die Gleichungen zweiter Ordnung linear sind, lassen sich diese anderen Effekte zweiter Ordnung getrennt berechnen und dem Wirbeleffekt überlagern, womit eine vollständige Theorie zweiter Ordnung zu erhalten ist.

Аннотация—Изучалось влияние внешней завихренности в области критической точки осесимметричного тупого тела на ламинарный пограничный слой сжимаемой жидкости. Рассматривается случай, когда завихренность создается криволинейным головным скачком уплотнения, образуемым этим телом, перемещающимся со сверхзвуковой скоростью. Используемый метод решения принадлежит Ван Дюку и состоит в решении уравнений пограничного слоя первого порядка (или Прандтля) и затем решении уравнений второго порядка для эффекта завихренности, где под вторым порядком подразумевается второй порядок в параметре возмущения, связанном с обратным квадратным корнем из числа Рейнольдса.

Первым шагом в решении как уравнений первого порядка, так и уравнений второго порядка для взаимодействия завихренностей является сведение дифференциальных уравнений в частных производных к обыкновенным дифференциальным уравнениям путем разложения в ряд Блазиуса. Затем полученные обыкновенные дифференциальные уравнения интегрируются численно. Приведены результаты интегрирования уравнений для первых двух членов ряда в случае уравнений первого порядка и для одного члена ряда для уравнений второго порядка. Эти данные получены для целого ряда отношений температуры на стенке к температуре критической точки у вершины. Данные, полученные путем интегрирования этих уравнений, сравниваются с теоретическими и экспериментальными данными других авторов. Найдено хорошее соответствие с теоретическими данными некоторых других авторов в случае, когда градиент давления, возникающего в результате взаимодействия вихрей, полагается равным нулю. Однако установлено, что вообще такое упрощение не допустимо. Установлено также соответствие с некоторой совокупностью экспериментальных данных, но это сравнение не имеет большого значения, т.к. приведенный анализ включает только эффект завихренности и не включает какие-либо другие эффекты второго порядка, которые могут быть столь же важны. Однако, поскольку уравнения второго порядка являются линейными, другие эффекты второго порядка можно рассчитать отдельно и сравнить с эффектом завихренности для получения полных данных в теории, учитывающей эффекты второго порядка.